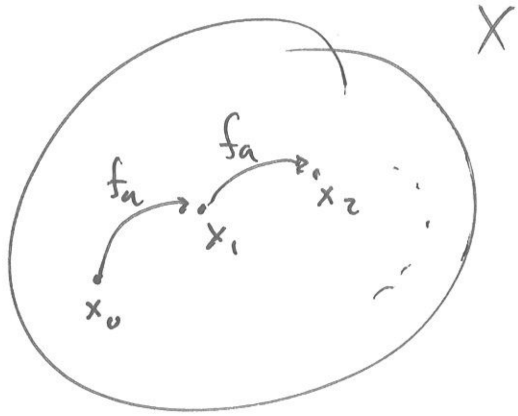
USP 2012 I

A dynamical system is a map

$$f_a : X \to X$$



X "state space"

f_a "evolution rule"

Usually dyn. sys. depend on a parameter a

The orbit of $x \in X$:

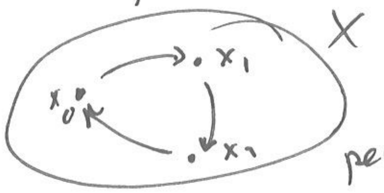
$$Orb(x) = \{ f^n(x) \mid n \geq 0 \}.$$

x is the initial state of the system and the orbit $Orb(x)$ describes the future.

—//—

What does it mean to understand a dyn. sys? The least you want to be able to do is

a) Predict future (long term / short term and quality).



period 3.

(eventually cyclic for example).

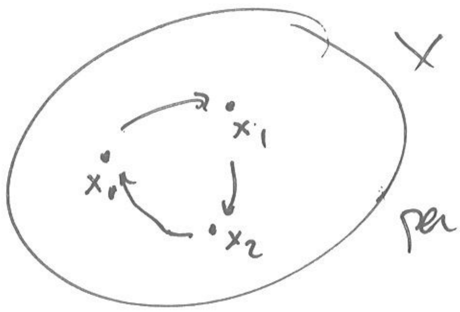
b) Parameter dependence. A dyn. sys is supposed to be a model of some physical system. How should one choose the parameter?

—//—

Problems / basic notions

* Periodic orbit: $\exists n \geq 0 \quad f^n(x) = x$

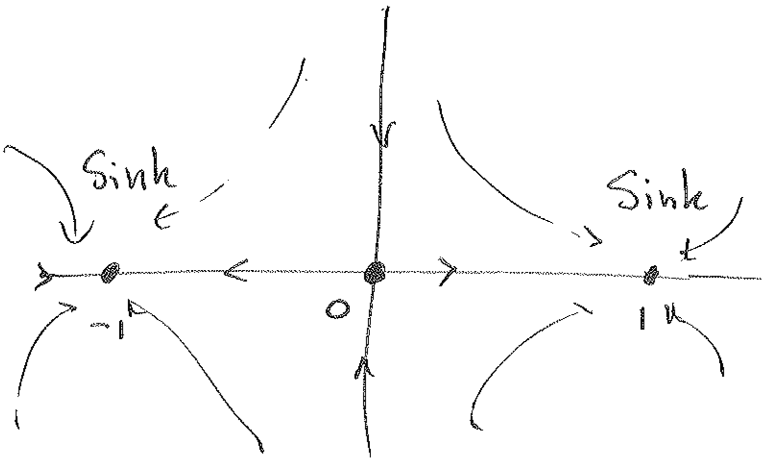
  minimal ~~n~~ n is the period.



period 3

Sink

* Periodic attractor: x periodic orbit with period n and there exist a ngh. $U \ni x$ s.t. $d(f^t(y), f^t(x)) \longrightarrow 0 \quad \forall y \in U$.



-1 and 1 are periodic attractors of period 1

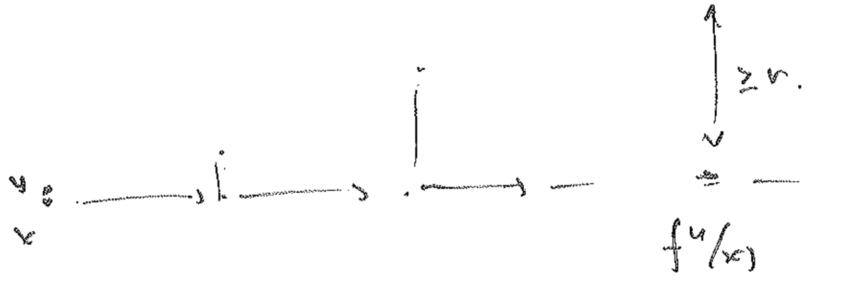
0 is not a periodic attractor (some points escape). It is called a saddle.

There is an extensive classification of periodic orbits. The simplest type are sinks / periodic attractors.

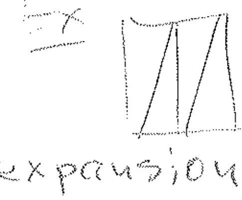
* Sensitivity to initial condition:

$\forall x \, \exists r>0 \, \forall \varepsilon>0 \, \exists \, y \in B_\varepsilon(x)$ and $n>0$ s.t.

$$d(f^n(x), f^n(y)) \geq r.$$



"Chaos"



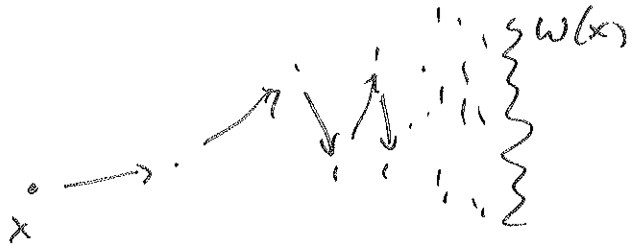
Rmk: if a system has S. to I.C it can not have periodic attractors

Rmk: if X is compact. the orbits $Orb(x)$ are going to be complicated, the will accumulate at ~~complicated~~ sets. with complicated dynamics. ("Chaos").

Rmk: prediction of the future is difficult/impossible in a chaotic system

* Limit set of x

$$\omega(x) = \bigcap_{n \geq 0} \overline{Orb(f^n(x))}$$



* Attractor $A \subset X$ is an attractor

Manifold with Lebesgue measure

if there exists $B_A \subset X$ with $\lambda(B_A) > 0$

s.t.

$$A = \omega(x) \quad x \in B_A .$$

B_A is the basin of attraction.

<u>Rmk</u>: it is <u>not</u> clear at all that systems do have attractors. It could be that each point has it own limit behavior.

* <u>Parameter Sensitivity</u>: There are many examples of ~~a~~ families $f_a: X \to X$, $a \in \mathbb{R}^n$ with sensitive parameter dependence.

For $a = 0$: f_a has Sensitivity to initial conditions / Chaotic.

$\exists |a| < 1$: f_a has ~~auto~~ periodic attractors.

<u>Rmk</u>: Dealing with a physical system Modelled by a dyn. system $f_a: X \to X$ it is very difficult to adjust the parameter.

Plan

Going back to the question ~~A~~ "What does it mean to understand a Dyn. Sys"

A suggestion to attack this problem is to concentrate on long term behavior:

Describe attractors &

- topological properties
- geometrical properties
- probabilistic properties
- parameter dependence.

This is an extremely involved project. So far it has let to ~~p~~ fascinating discoveries and very powerfull tools.

Rmk: These discoveries and tools will/mig be instrumental in solving actual physical/Industrial problems.

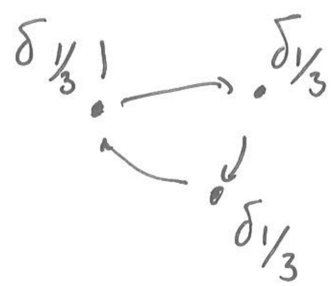
Ergodic Theory

Often an attractor $A \subset X$ carries an invariant measure ν which describes the distribution of orbits along the attractor: Let $U \subset X$ open.

$$\lim_{n \to \infty} \frac{1}{n} \# \{ i \leq n \mid f^i(x) \in U \} = \nu(U).$$

for a.e. $x \in B_A$.

Ex: point masses along periodic attractor.



,

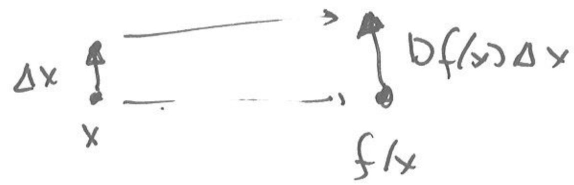
$f_{p \in P}$.

Ergodic Theorem: Let ~~$\varphi \in L^1(X, \nu)$~~. $\varphi \in C^0(X)$.

$$\frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \longrightarrow \int \varphi \, d\nu \qquad \text{a.e. } x \in B_A$$

Hyperbolicity

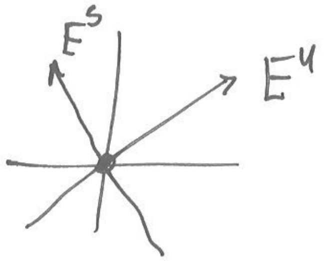
Sensitivity is associated to expansion.



with $|Df(x)\Delta x| > |\Delta x|$

Example: The matrix $\begin{pmatrix}2&1\\1&1\end{pmatrix}$ induces a map $f: T^2 \longrightarrow T^2$ with $Df(x) = ($

Eigenvectors



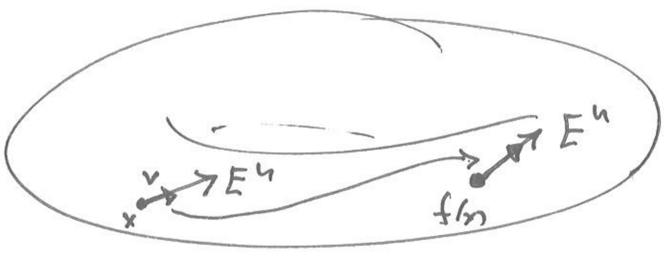
: $|Df(x)v| \geq \lambda^u |v|$ ve

$|Df(x)v| = \lambda^s |v|$ ve

$\lambda^u > 1 \quad \lambda^s < 1$

Moreover $Df(x)E^u = E^u$. (invariant direction of f).

The sensitivity occurs along the E^u direction.



A system $f: M \to M$ (M manifold)
f diffeo is called hyperbolic if
there exist a continuous splitting of the
tangent bundle:

- $\forall x \in M \quad T_x M = E^u_x \oplus E^s_x$

- $Df(x) E^u_x = E^u_{f(x)}$, $Df(x) E^s_x = E^s_{f(x)}$

- $\exists\ |\lambda^u| > 1 \quad |\lambda^s| < 1$ s.t.

$$|Df^n(x) v| \geq (\lambda^u)^n |v| \qquad \forall v \in E^u_x$$
$$|Df^n(x) v| \leq |\lambda^s|^n |v| \qquad \forall v \in E^s_x$$

<u>Rmk</u>: hyperbolic system are well
understood. in the sense of the
<u>Plan</u>. Unfortunately there are
not so many of them.

Fortunately, every system has a measure theoretical form of hyperbolicity

Thm (Oseledec): $f: M \to M$ diffeo.

ν ergodic measure ($\lim \frac{1}{n} \sum \varphi(f^i) \to \int \varphi d\nu$)

Then $\exists\ d_1, d_2, \dots, d_s \in \mathbb{N}$ with $\sum d_i = \dim M$

and $\exists\ \lambda_1, \lambda_2, \dots, \lambda_s \in \mathbb{R}$ s.t.
(Lyaponov exponents)

for almost every $x \in M$.

- $T_x M = E_x^1 \oplus E_x^2 \oplus \cdots \oplus E_x^s$ (measurable dependence)
- $Df(x)\, E_x^i = E_{f(x)}^i$
- $\lim\limits_{n\to\infty} \frac{1}{n} \ln |Df^n(x) v| = \ln|\lambda_i|$ $\quad \forall\, v \in E_x^i \setminus \{0\}$

$$\operatorname{Rank}(f,\nu) = \sum_{|\lambda_i| \geq 1} d_i$$ (dim of unstable directions)

Systems with rank ≥ 2 are still poorly understood.

This course will concentrate on the simplest non-trivial systems with rank $= 1$ in dimension 1 and 2.

This mini-course concentrates on Rank = 1 dynamics in 1 and 2 dim. dynamics. with the simplest non-trivial topological structure. Although the simplicity of the situation we will encounter fascinating phenomena and power full tools which are at the core of more general theories, theories for general ~~top~~ situations.

The maps we will consider are at transition to chaos:

$$f_t : X \to X \qquad t \in (-\varepsilon, \varepsilon)$$

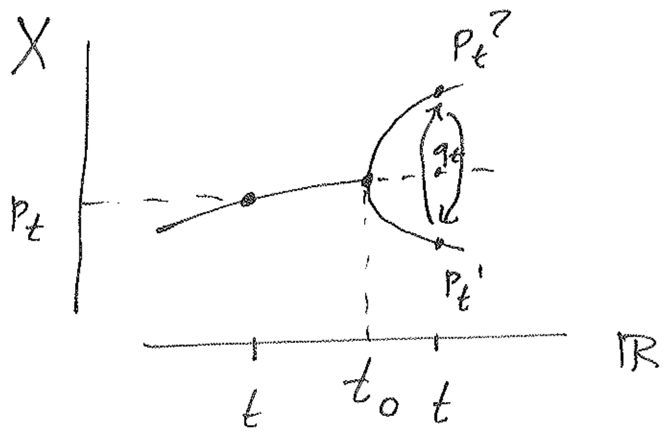
$f_t, t<0$: no - sensitivity to initial condition (periodic attractors).

$f_t, t>0$: sensitivity. (Chaos). (on subsets)

This is not a theorem but it seems that in Rank 1 dynamics the transition to chaos occurs along so called period doubling cascades: when the system f_t approaches the boundary of chaos there will be a sequence of moments when the periodic attractor doubles its period.

⊛ Period Doubling Bifurcation

$$f_t : X \longrightarrow X$$

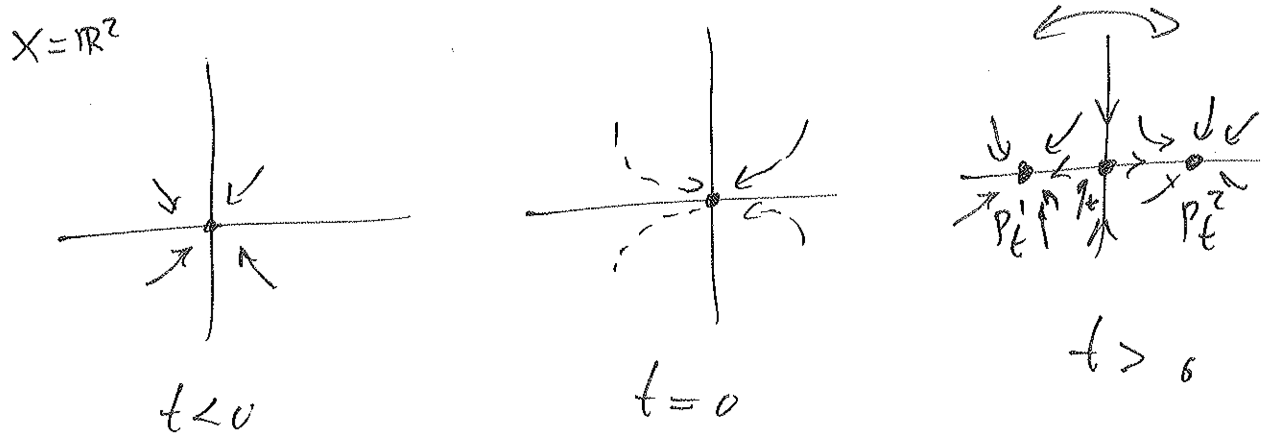


$t < t_0$: p_t ~~is~~ attracting fixed point.

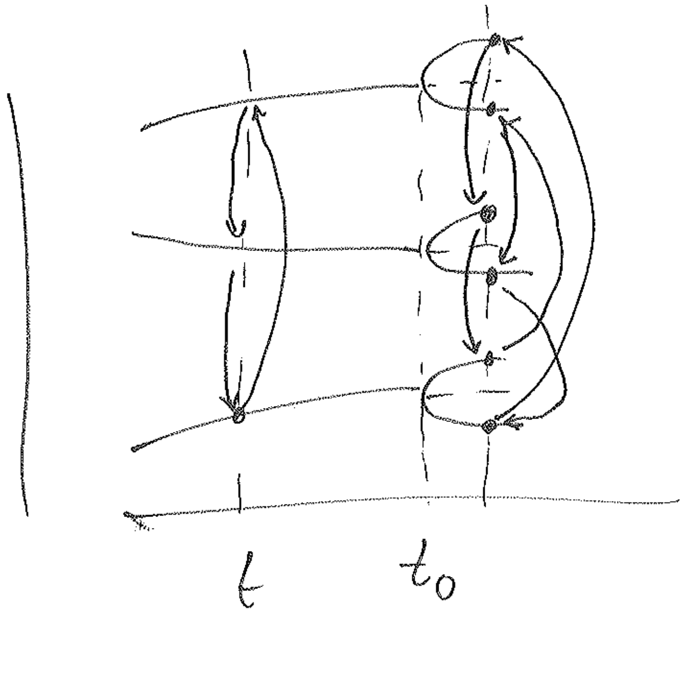
$t > t_0$: $\{p_t^1, p_t^2\}$ attracting periodic orbit of period 2.

q_t is the "continuation" of p_t, $t<0$

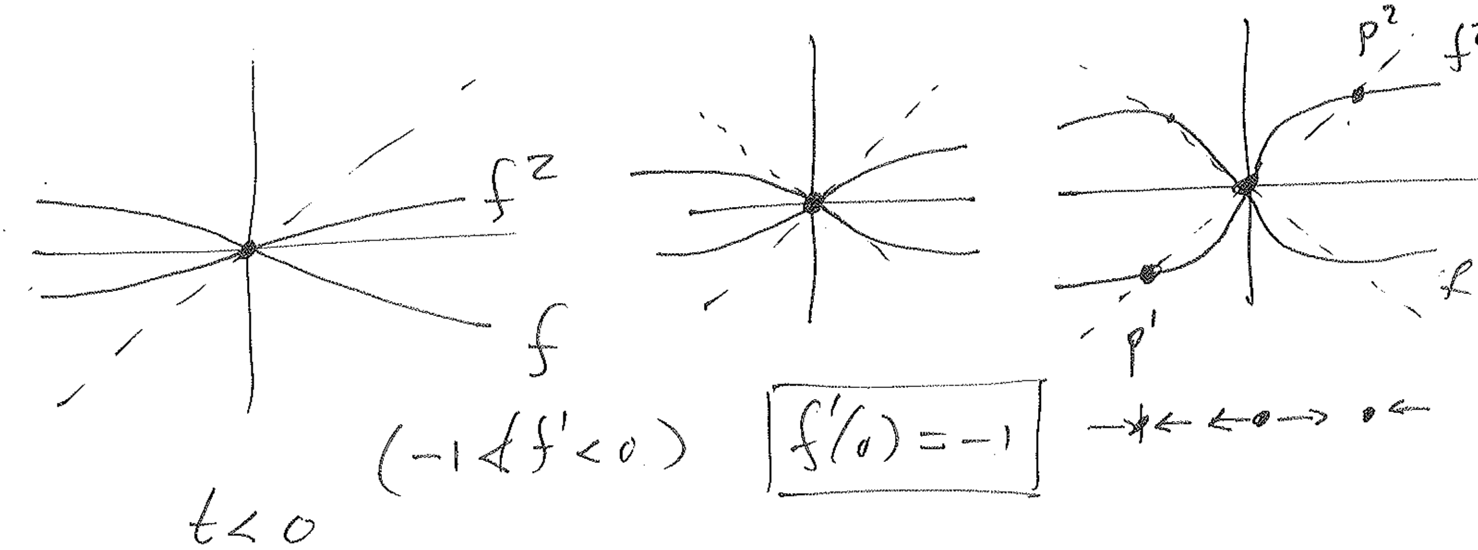
q_t is not attracting anymore, it will have at least 1 unstable direction



The figure illustrates the period doubling bifurcation of a fixed pt. Similarly, one has pd bif of arbitrary period. For <u>example period 3</u>

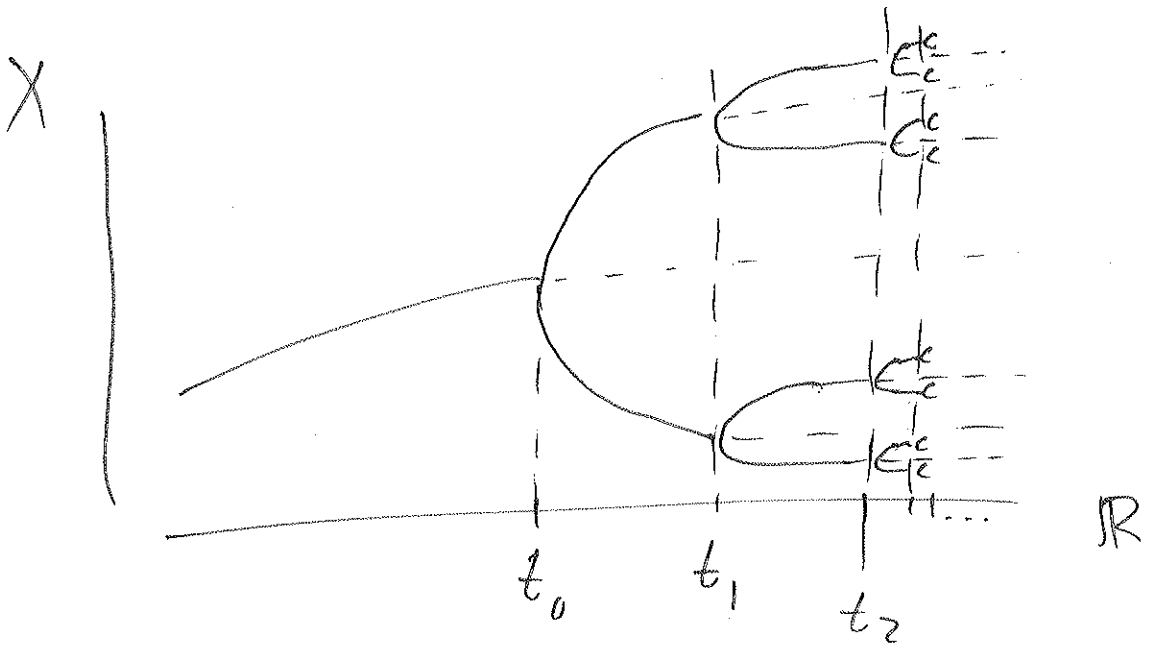


1 Dim example $f_t : \mathbb{R} \to \mathbb{R}$.



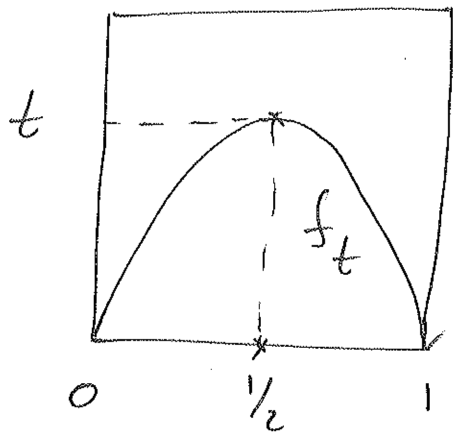
⊛ Period Doubling Cascade

PD cascade is a sequence of PD bifurcations. $t_n \to t_\infty$



⊛ ∂Chaos (Boundary of Chaos).

in 1 dim



$f_t : [0,1] \longrightarrow [0,1] \quad C^\omega$

- $Df_t(\frac{1}{2}) = 0$ critical point.
- $D^2 f_t(\frac{1}{2}) < 0$ non-degenerate critical point.
- $f_t(\frac{1}{2}) = t$.

Example: $f_t(x) = 4tx(1-x)$.

Thm: If f_{t_∞} is at the boundary of Chaos
~~(Misiurewicz)~~
Block & Hart

no-sea sensitivity

t_∞

the there exists a period Cascade

$$t_n \longrightarrow t_\infty.$$

Coullet–Tresser / Feigenbaum discovered fascinating and fundamental geometrical properties of PD Cascades.

Parameter Universality

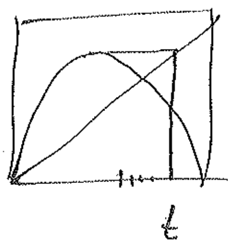
$$\frac{t_{n+1}-t_n}{t_n-t_{n-1}} \longrightarrow \frac{1}{4.669\ldots}$$

(4.669… for <u>all</u> families f_t !!!)

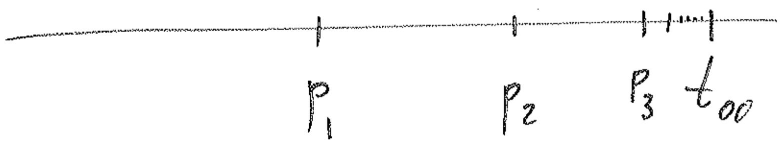
Phase Space Universality

At t_∞ f_{t_∞} will have periodic orbits P_n of period 2^n. These are the orbits left after the PD bif (q').

In each orbit P_n choose p_n to be the right most then



$$p_n \longrightarrow t_\infty$$



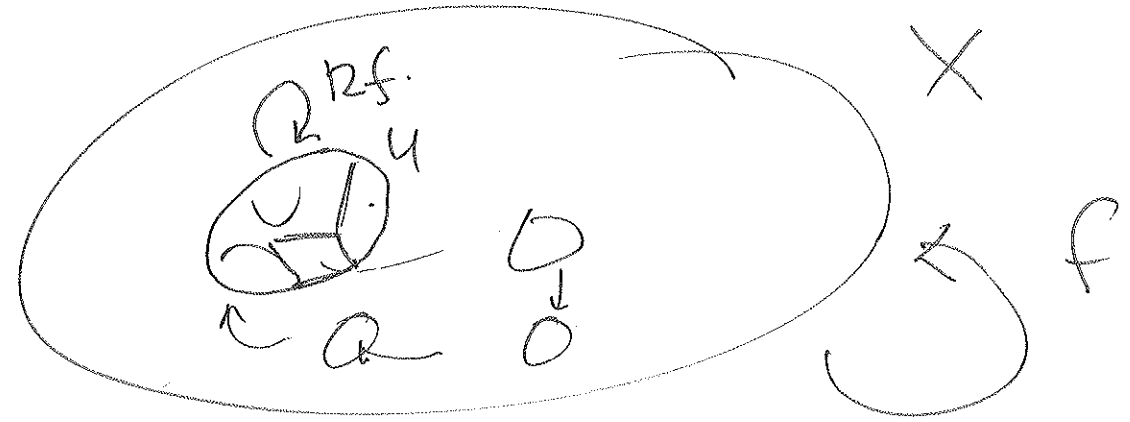
$$\frac{p_n - t_\infty}{p_{n+1}-t_\infty} \longrightarrow (2.66\ldots)^2$$

($(2.6.)^2$ for <u>all</u> families f_t !!!)

Renormalization

Renormalization is like a microscope for dynamics. It allows to consider the dynamics on smaller scales.

Rf = first return map to appropriatly chosen domains



Let U be a small domain containing the area of our interest.

$x \in U$: usually there exists $n_x > 0$ (minimal) such that $f^{n_x}(x) \in U$.

$$Rf : U \longrightarrow U$$

$$Rf(x) = f^{n_x}(x).$$

Rf forgot global dynamics in X but knows all what happens on the smaller scale U.

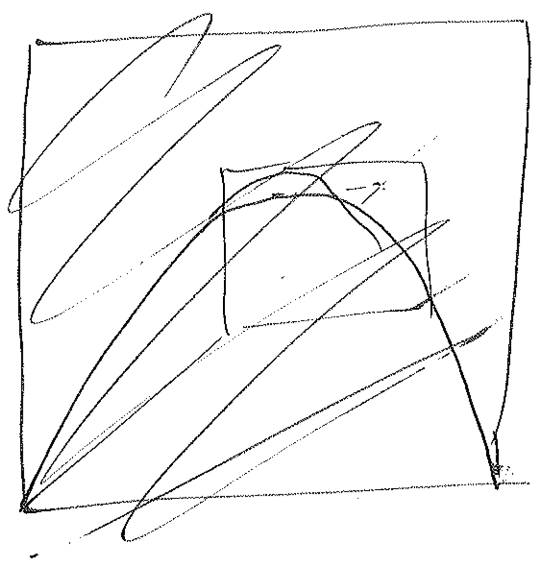
<u>Rmk</u>: n_x is usually piecewise constant. Rf has usually many branches.

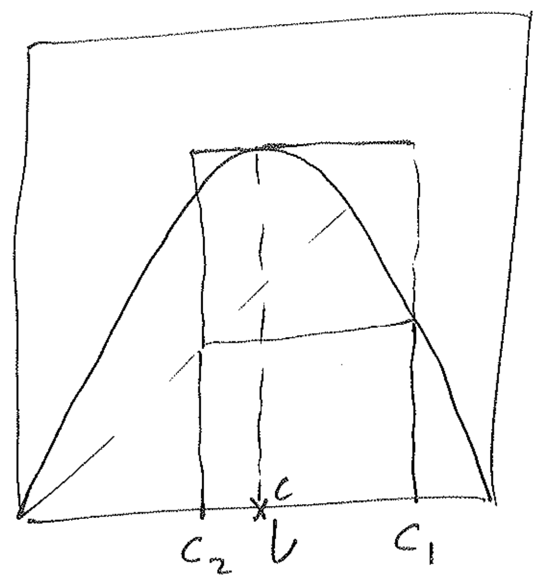
This Renormalization operation was introduced by Coullet & Tresser to understand the maps at the boundary of Chaos.

PD Renormalization in 1 Dim

Let $f : [0,1] \longrightarrow [0,1]$ be at the boundary of chaos.

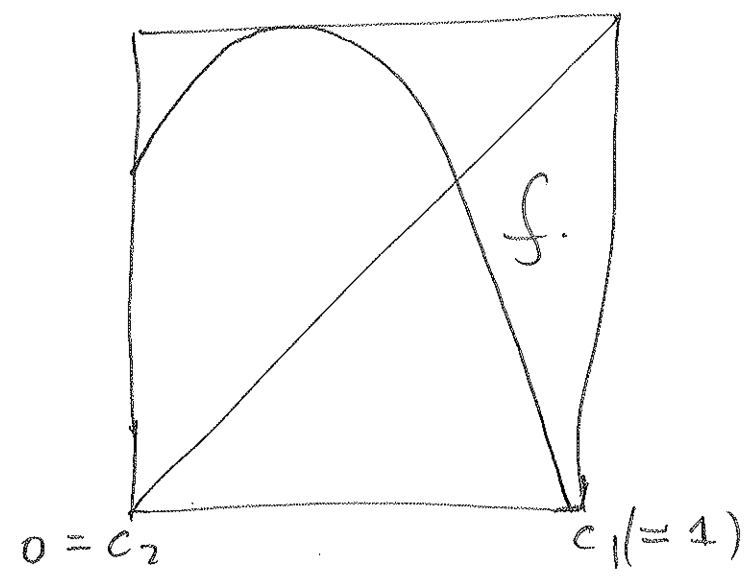
$c_n = f^n(c)$



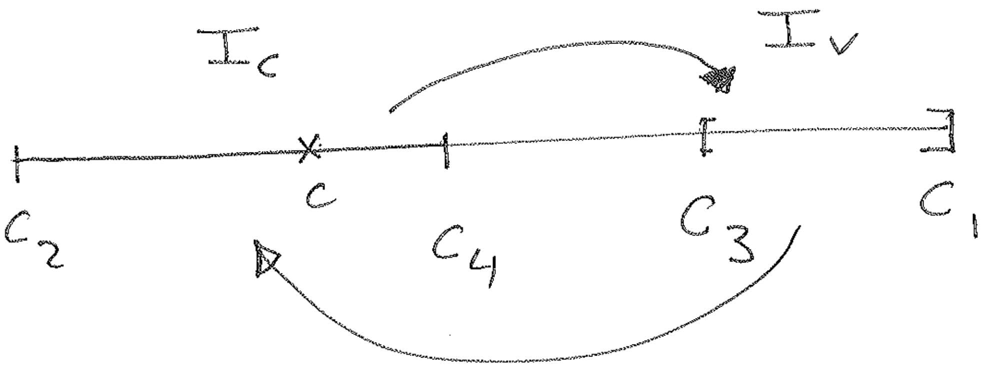


It is more convenient to restrict the map to the interval $[c_2, c_1]$ where $c_1 = f(c)$, $c_2 = f(c_1)$, c critical pt.

Observe, all orbits enter $[c_1, c_2]$ and never leave.



The maps at the boundary of chaos have the property.



$$c_2 < c < c_4 < c_3 < c_1$$

Let $I_c = [c_2, c_4]$

$I_v = [c_3, c_1]$.

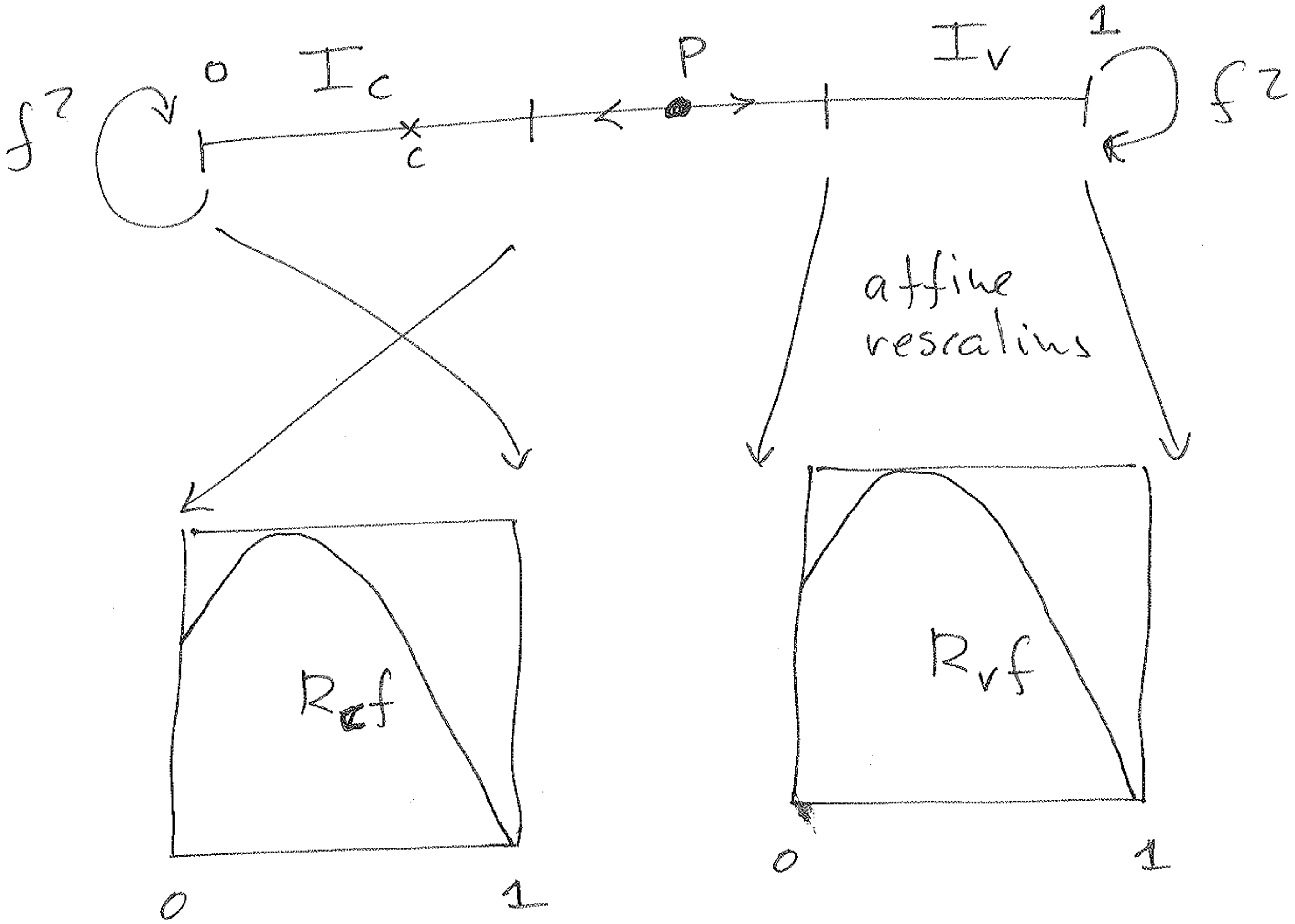
<u>Observe,</u> f exchanges I_c and I_v.

$$\left.\begin{aligned} f(I_c) &= I_v \\ f(I_v) &= I_c. \end{aligned}\right\} \text{Renormalizable}$$

Natural thing to do:

Renormalize to I_c : $R_c f$ or

Renormalize to I_v : $R_v f$



$R_v f$ = affine rescaling to $f^2|I_v$.

$R_c f$ = — — ———————— $f^2|I_c$.

<u>Rmk</u>: between I_c and I_v there is an expanding fixed pt. Except for p all orbits will eventually land in $I_c \cup I_v$ and stay there.

Each orbit, except Orb(p), will eventually be jumping forth and back between I_c and I_v.

If you want to study the part of an orbit in I_c forget f and use $R_c f$. instead. Similarly, $R_v f$ for dynamics in I_v.

$R_c f, R_v f$ describe dynamics on smaller scale I_c, I_v. resp.

—//—

For maps in the boundary of chaos the process ~~repeats~~. repeats.

- except for the periodic orbit in $\mathcal{C}_n$ every orbit will eventually land in $\mathcal{C}_{n+1}$

Let $\mathcal{O}_f = \bigcap \mathcal{C}_n$

<u>Thm</u>: • $\mathcal{O}_f$ is a Cantor attractor.

$\omega(x) = \mathcal{O}_f$ a.e. x. (except the periodic orbits).

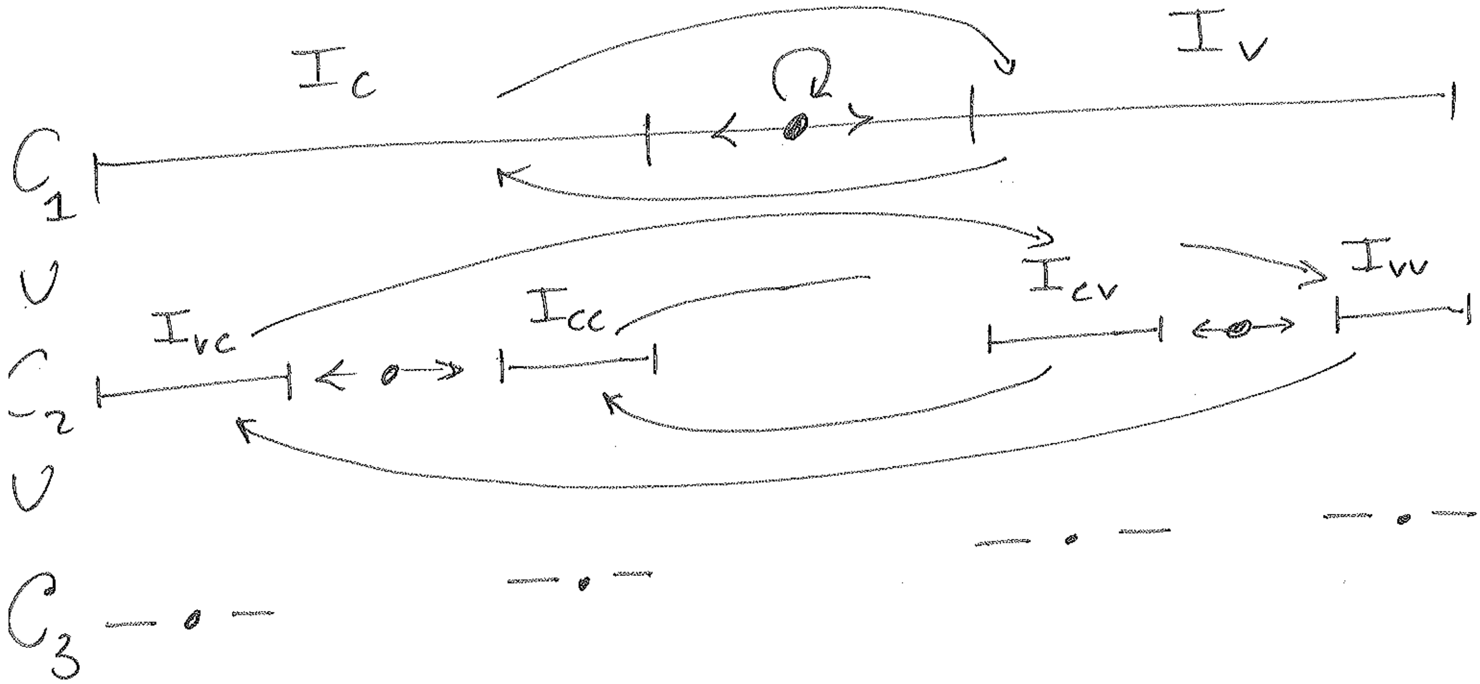
• $\mathcal{O}_f$ has a unique measure μ which describes the dynamics.

$$\frac{1/2}{I_c} \qquad\qquad \frac{1/2}{I_v}$$

$$\frac{1/4}{I_{vc}} \quad \frac{1/4}{I_{cc}} \qquad\qquad \frac{1/4}{I_{cv}} \quad \frac{1/4}{I_{vv}}.$$

"each orbit will spend a 1/4 of its time in I_{cv}"

So the topogical situation for maps on the boundary / maps th- which are infinitely renormalizable is

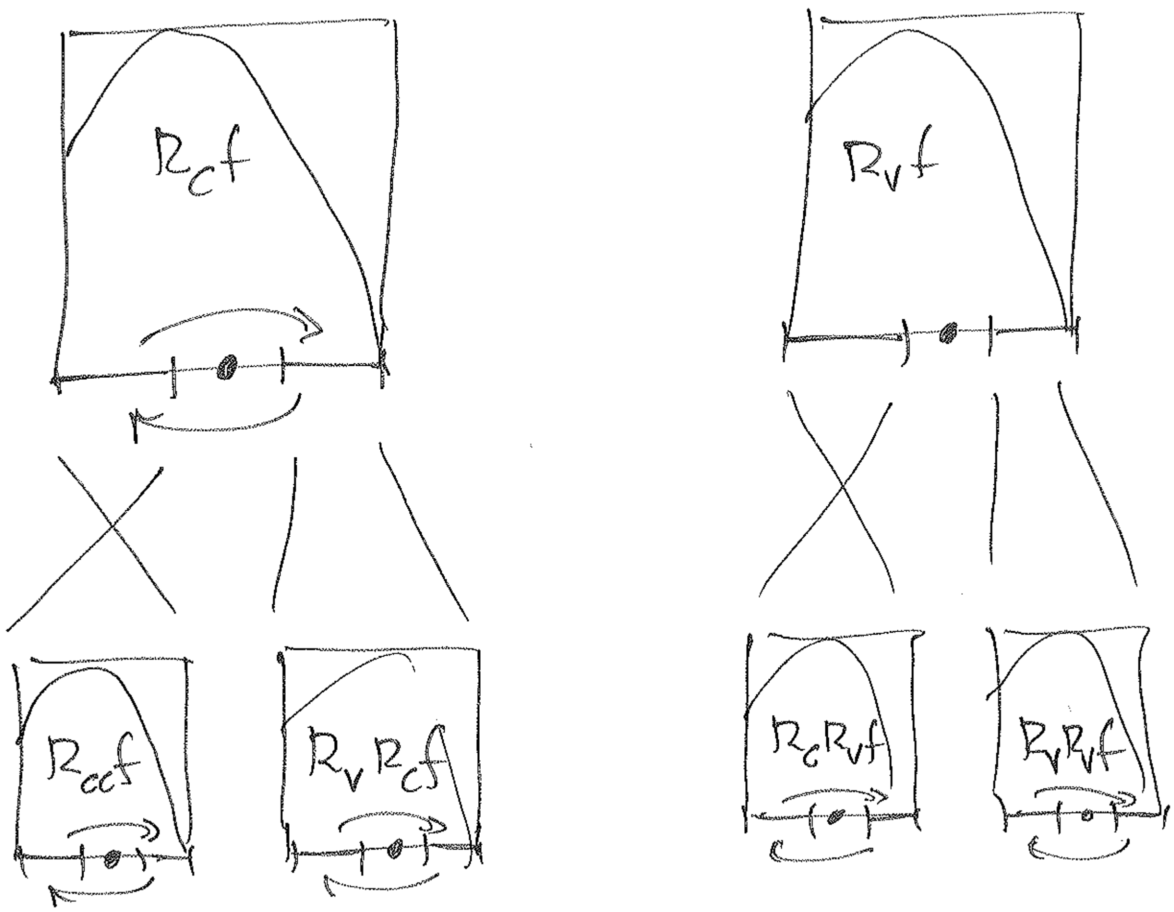


There is a nested sequence of collections of intervals

$$\mathcal{C}_0 = \{[0,1]\} \supset \mathcal{C}_1 \supset \mathcal{C}_2 \supset \cdots$$

- $\#\mathcal{C}_n = 2^n$
- each $\mathcal{C}_n$ contains a periodic orbit of period 2^n
- f permutes the intervals in $\mathcal{C}_n$.

$R_c f$ and $R_v f$ are renormalizable again



etc.

Conclusion: Renormalization

explain the dynamics of maps on the boundary of chaos

- topology
- Erg. Theory (μ)

- Geo theory
- parameter dependence

} Tomorrow