USP 2012 IV

2Dim Dymacis: (dissipative)

$$f_t : [0,1]^2 \longrightarrow [0,1]^2$$

$$0 < \det Df \ll 1 \quad \text{(dissipation / friction.)}$$

Consequence: The attractors have measure zero. Observe, let.

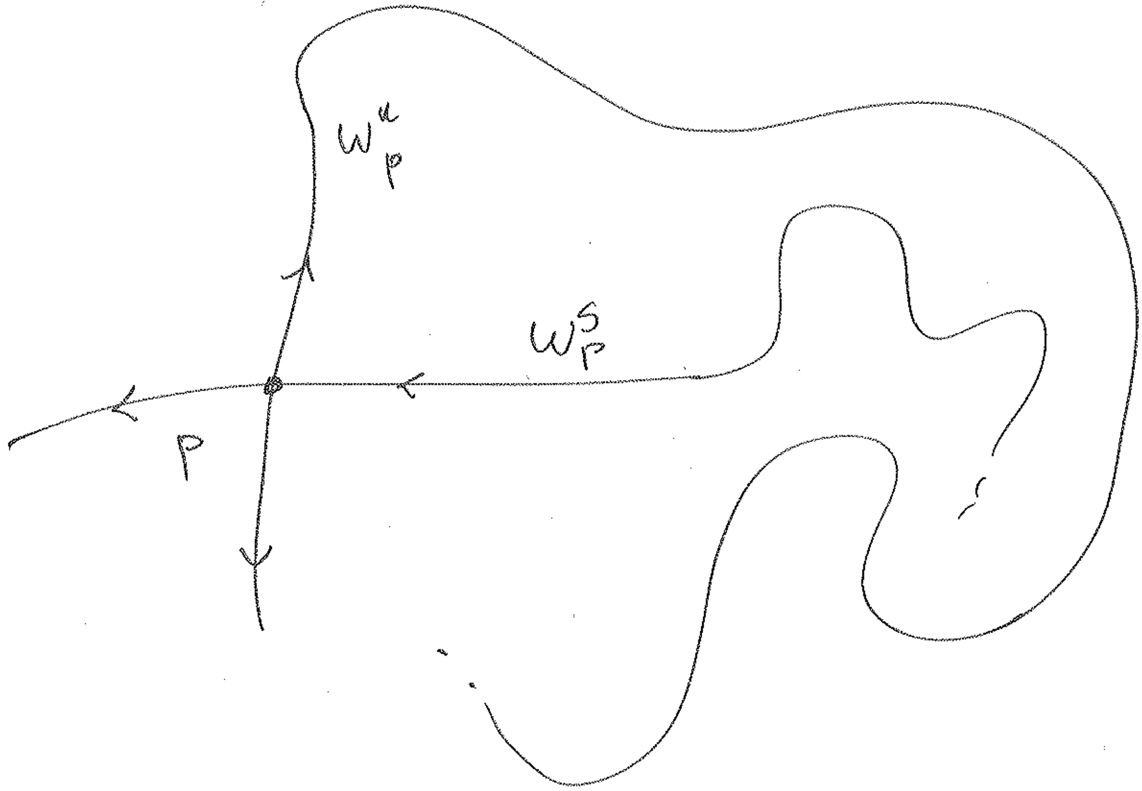
$$A_{Top} = \bigcap_{n \geq 0} F^n([0,1]^2)$$

the $|A_{top}| = 0$ and every attractor

$$A \subset A_{top}.$$

Rmk: $\quad |F(X)| = \int_X |\det Df| \ll |X|.$

Saddle points have stable and unstable manifolds.



$$W_p^s = \{x \mid F^n x \longrightarrow p\}.$$

$$W_p^u = \{x \mid F^{-n} x \longrightarrow p\}$$

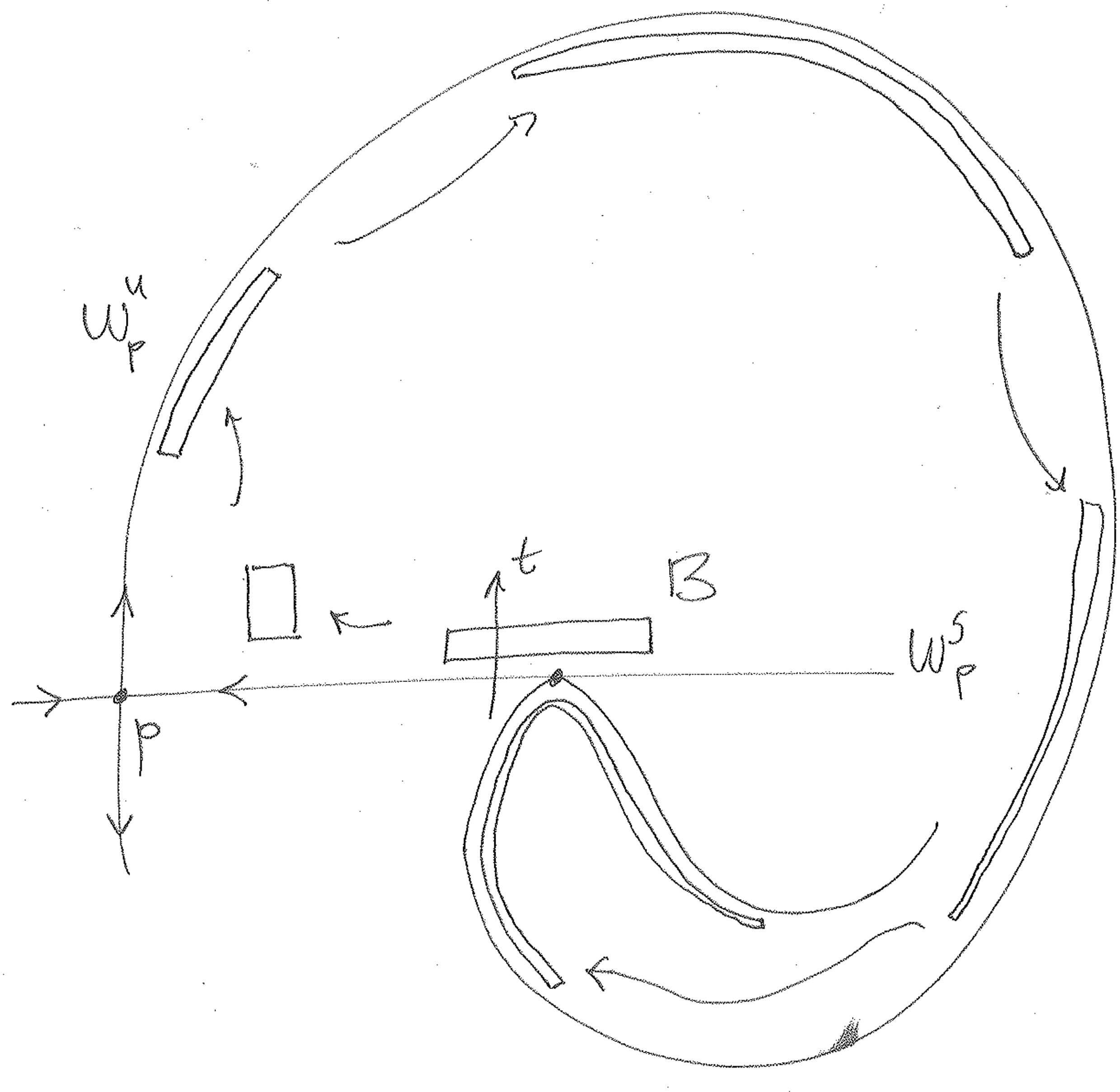
Important topological observation:

Topological changes in the family F_t are associated with tangencies between unstable and stable manifold of saddle orbits.

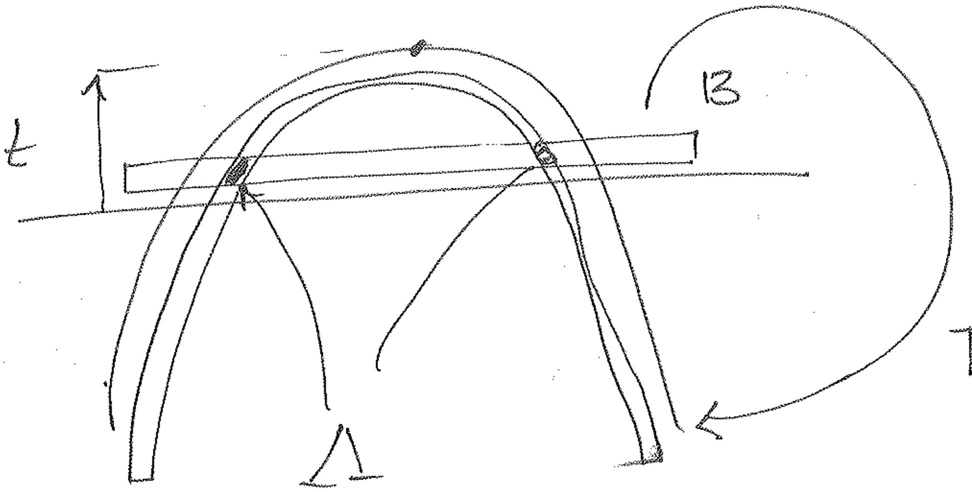
The simplest situation is a homoclinic tangency

$$W^u_p \pitchfork W^s_p$$

(a tangency between the manifolds of the same orbit).



<u>Rmk</u>: Say the tangency occurs at $t=0$ and for $t>0$



The in $B \cap F^N(B) \supset \Lambda$ there is a new set Λ which has sensitive dependence on initial cond.

By moving $t>0$, you create <u>new</u> chaos.

$$\boxed{\text{Homoclinic tangencies} \longleftrightarrow \text{creation of (new) Chaos}}$$

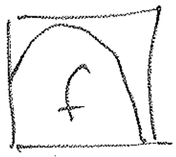
The map

$$F_t^N : B \longrightarrow F_t^N(B) \subset \mathbb{R}^2$$

is up to smooth coordinate change of the form. (Hénon-Map)

$$F_t : [0,1]^2 \longrightarrow [0,1]^2$$

$$F_t \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x) - \varepsilon(x,y) \\ x \end{pmatrix}$$

where $f \in \mathcal{U}$ unimodal: 

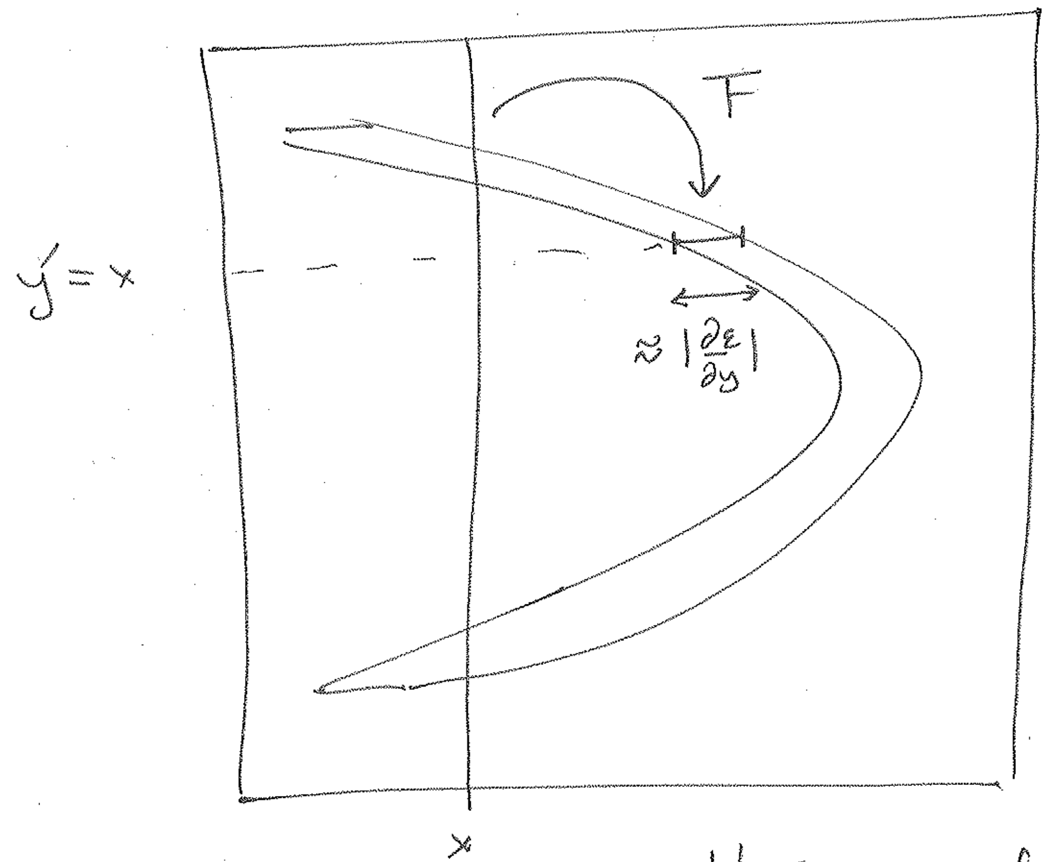
$$|\varepsilon| \ll 1.$$

Rmk

$$\det DF = \text{Jacobian of } F$$
$$= \operatorname{Jac} F = \frac{\partial \varepsilon}{\partial y}$$

Rmk: $|\varepsilon| \ll 1$: F is a small perturbation of a unimodal map.

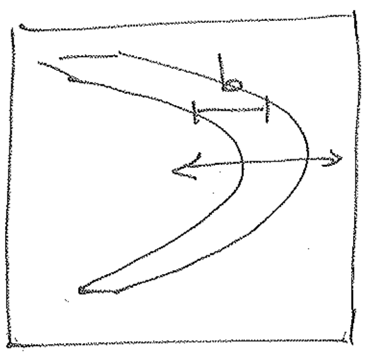
Rmk: Vertical lines are contracted and mapped horizontally.



Rmk: The classical Hénon-family.

$$F_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 - by \\ x \end{pmatrix}$$

$$\mathrm{Jac}\, F = b.$$

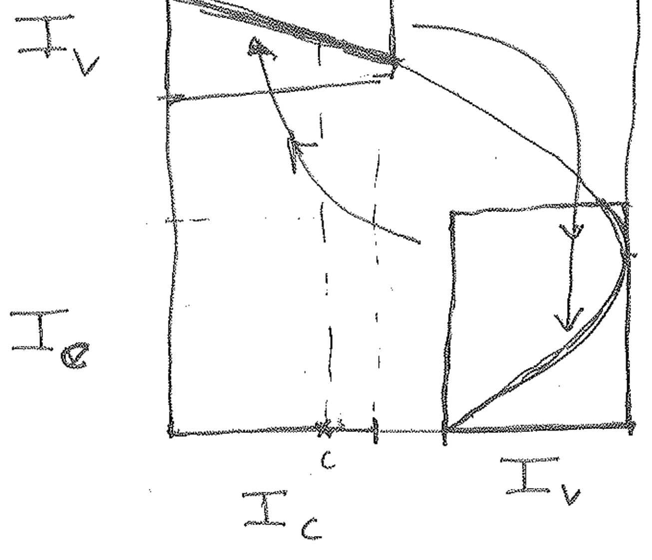


a variation.

Hénon Renormalization

The goal is to understand the dynamics at the boundary of chaos for strongly dissipative Hénon-maps. These maps are small perturbations of the unimodal situation. We expect PD renormalization to be a usefull tool.

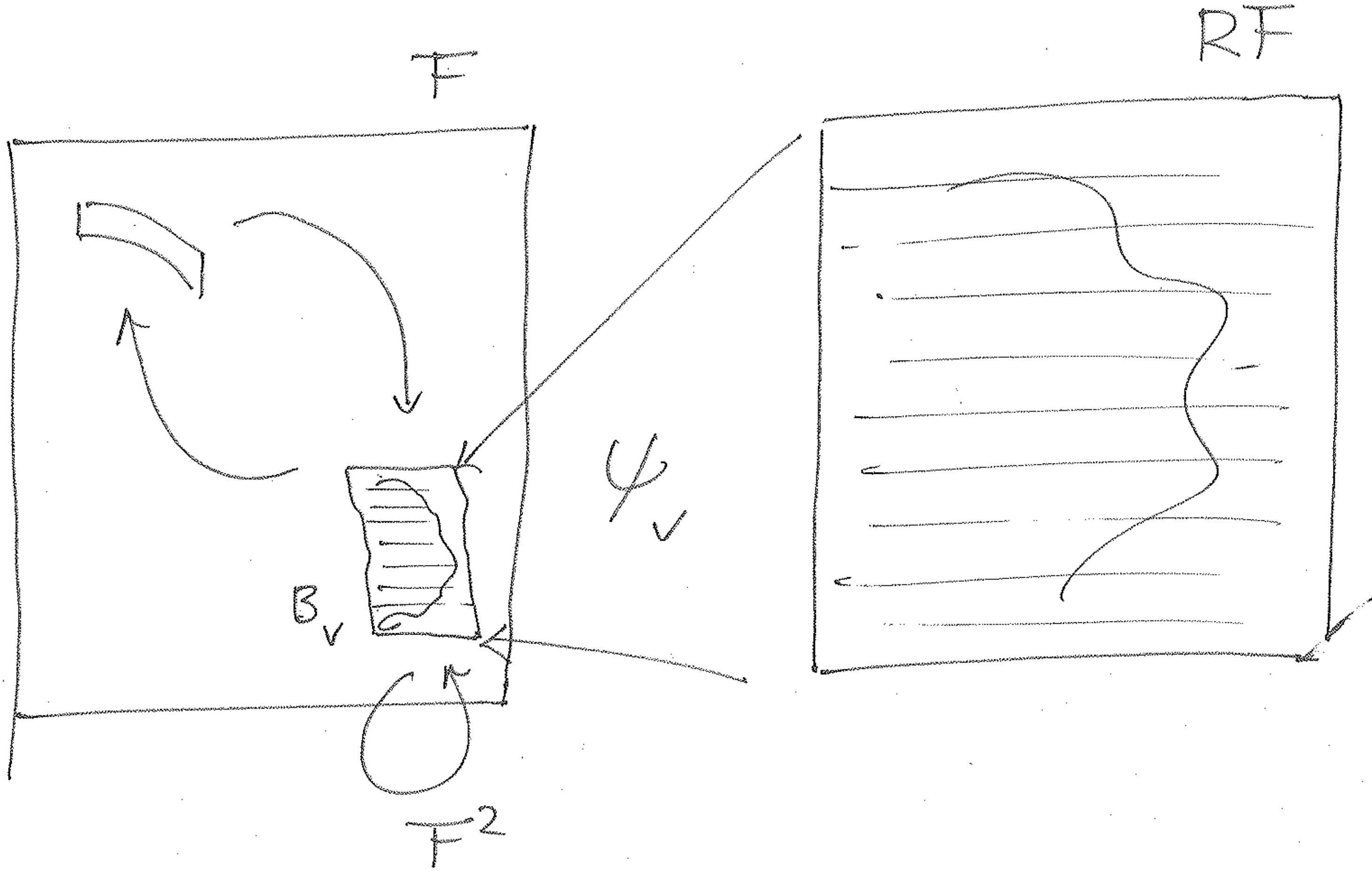
Example: let $f\ [0,1] \to [0,1]$ be renormalizable and $F = \begin{pmatrix} f(x) \\ x \end{pmatrix}$ $(\varepsilon \equiv 0)$.



$F(I_v \times I_c) \subset I_c \times I_v$

$F(I_c \times I_v) \subset I_v \times I_c$

In geral, F is renormalizable if there are two disjoint pieces B_c, B_v which are exchanged by F.

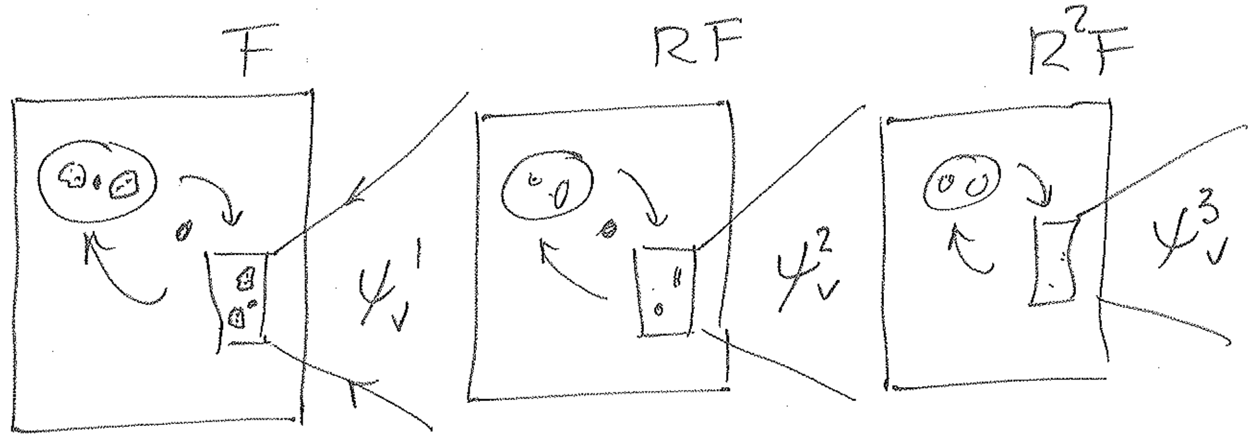


$$RF = \psi_v^{-1} \circ F^2 \circ \psi_v$$

Rmk: ψ_v is not affine. but it preserves horizontal lines. The form of F^2 is not Hénon: $F^2 = \begin{pmatrix} \sim\sim \\ f(x) - \varepsilon(x,y) \end{pmatrix}$

We need a non-affine coordinate change/rescaling ψ_v to bring F^2 back to Hénon shape. However, ψ_v has an explicitely defined expression. (~~uniqu~~ unique).

Let F be ∞-renormalizable.



$$\psi_c^1 = F\circ\psi_v^1 \qquad \psi_c^2 = RF\circ\psi_v^2$$

$$B_v^1 = \mathrm{im}\,\psi_v^1$$

$$B_c^1 = \mathrm{im}\,\psi_c^1 = F(B_v^1)$$

$$C_1 = \{B_c^1, B_v^1\}$$

$$C_2 = \{B_{cc}^2, B_{cv}^2, B_{vc}^2, B_{vv}^2\}$$ where

$$B_{vv} = \psi_v^1 \circ \psi_v^2([0,1]^2)$$

$$B_{cv} = \psi_c^1 \circ \psi_v^2([0,1]^2)$$

$$B_{vc} = \psi_v^1 \circ \psi_c^2([0,1]^2)$$

$$B_{cc} = \psi_c^1 \circ \psi_c^2([0,1]^2).$$

In general $\omega = \omega_1 \omega_2 \cdots \omega_n$

$$\psi_\omega = \psi_{\omega_1}^1 \circ \psi_{\omega_2}^2 \circ \cdots \circ \psi_{\omega_n}^n.$$

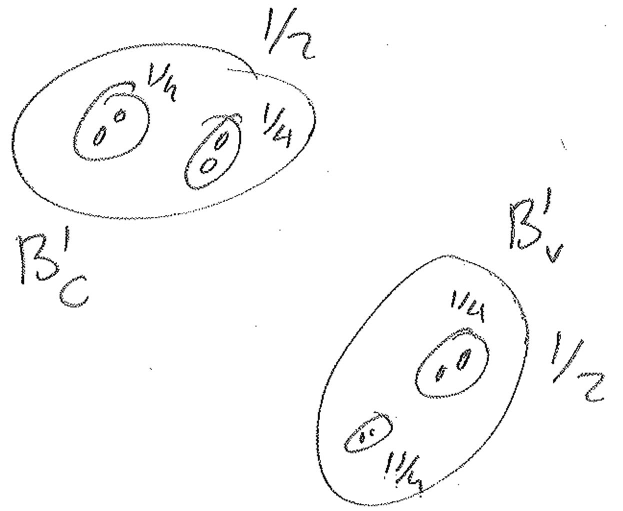
$$B_\omega = \operatorname{im} \psi_\omega.$$

$$C_n = \{ B_\omega \mid |\omega| = n \}.$$

$$C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$$

$$O_F = \bigcap C_n.$$

3) The dynamics on $\mathcal{O}_F$ is topologically the same as on $\mathcal{O}_{f_*}$ (1D). In particular $\mathcal{O}_F$ carries a measure μ.



4) The average Jacobian

$$\boxed{b_F \overset{\text{def}}{=} e^{\int \ln \operatorname{Jac} F \, d\mu}}.$$

$\lambda_1 = 1$ $\quad \lambda_2 = b_F < 1$ (Lyapunov exponents)

(Rank = 1).

5) $0 <$ Hausdorff dimension $\mathcal{O}_F < 1$

<u>Rmk</u>: b_F is going to play a big top, geo, measure theoretical role!!

The topological properties of the renormalization cycles $\mathcal{C}_n$ is as in the 1 Dim case.

- $\# \mathcal{C}_n = 2^n$
- each $\mathcal{C}_n$ contains a unique periodic orbit P_n of saddle type with period 2^n.
- F permutes the pieces of $\mathcal{C}_n$.

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Group working on Hénon-Rwhich obtained the following results
A. de Carvalho, P. Hazard, S. Chandramoulli, M. Lyubich, M, Y.W. Nam.

<u>Thm</u>: 1) $\mathcal{O}_F$ is a Cantor set.

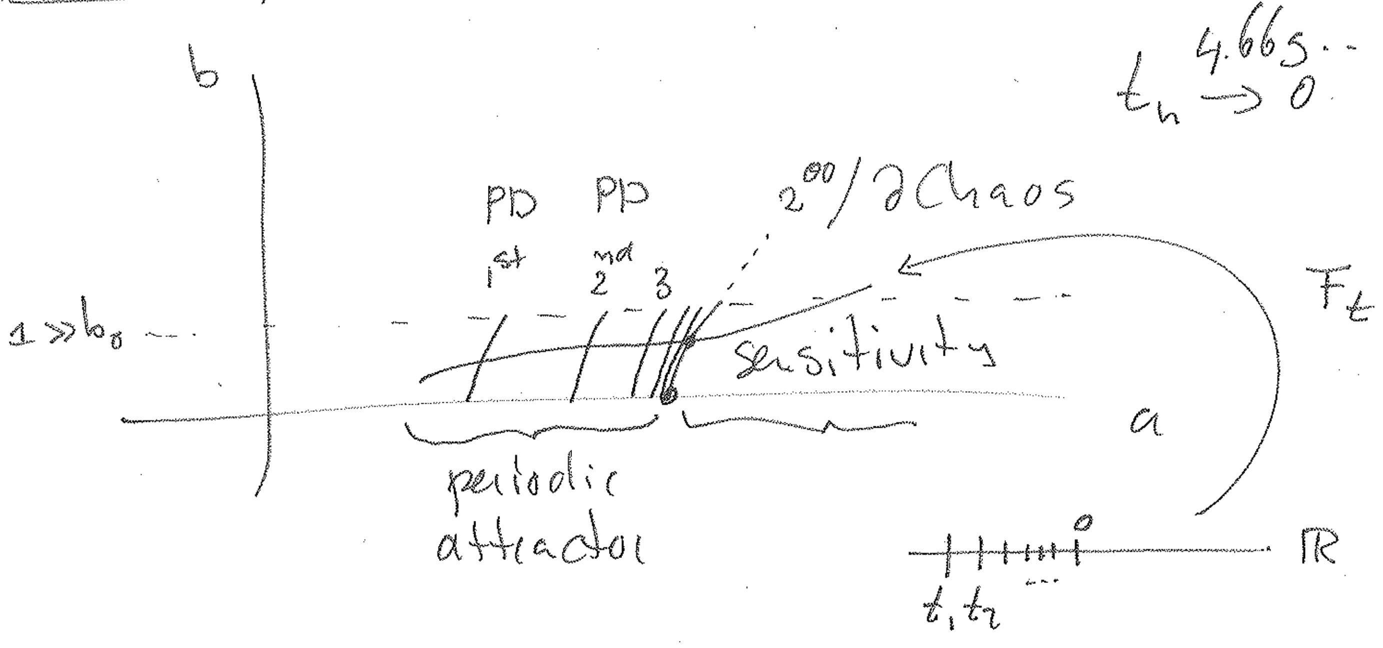
2) $\mathcal{O}_F = \omega(x)$ a.e. x Lebesgue.

$(x \notin \cup W^s_{P_n})$.

More good but expected news

<u>Thm</u>: In the space of strongly dissipative Hénon-maps the boundary of chaos is a codim 1 submanifold which consists of ∞-ren. Hénon-maps. Moreover transition to chaos occurs along a period doubling cascade with 4.669.. as rate.

<u>Example</u>: Classical Hénon Family

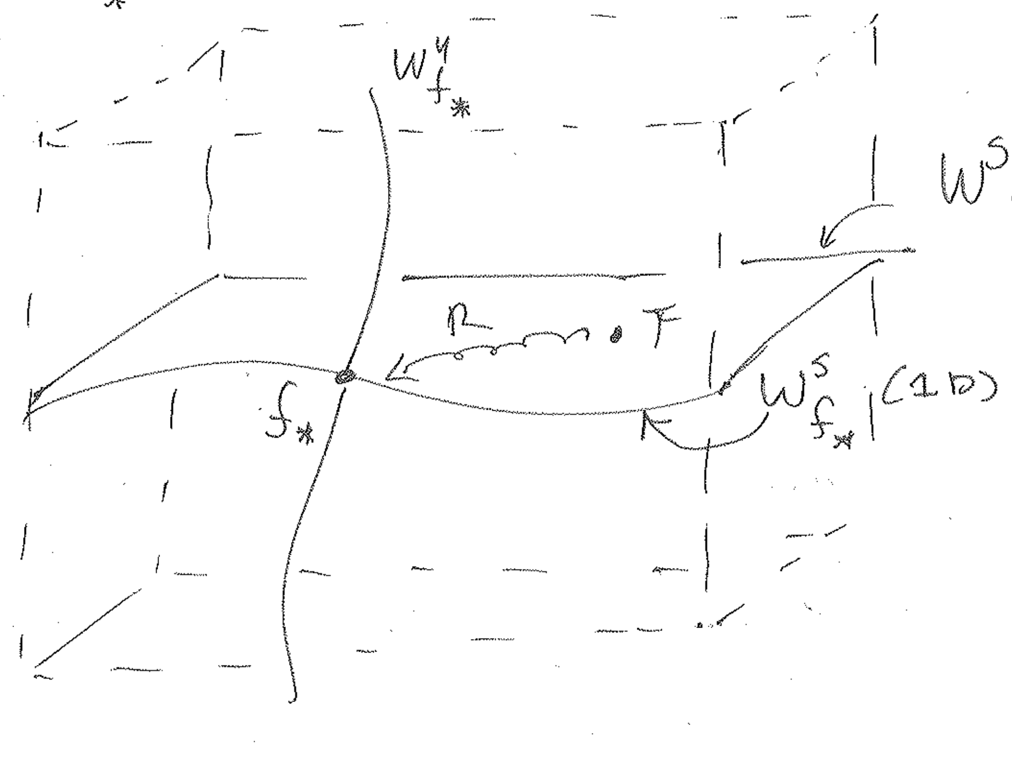


The theorem behind these results is the hyperbolicity of R.

<u>Thm</u>: On the space of strongly dissipative Hénon-maps ($|\varepsilon| \ll 1$) R is hyperbolic with fixed pt $F_* = \begin{pmatrix} f_*(x) \\ x \end{pmatrix}$ where f_* is the 1 Dim Ren. fixed pt.

$W^u_{F_*} = W^u_{f_*}$ $\quad \delta = 4.669\ldots$

$W^s_{F_*} = \{\infty\text{-ren}\}$ codim $= 1$.



Rmk: Collet & Eckmann & Koch introduced a Hénon-period doubling operators. Which also turned out to be hyperbolic. However, it was not suitable to study the geometry of $\mathcal{O}_F$.

Universality

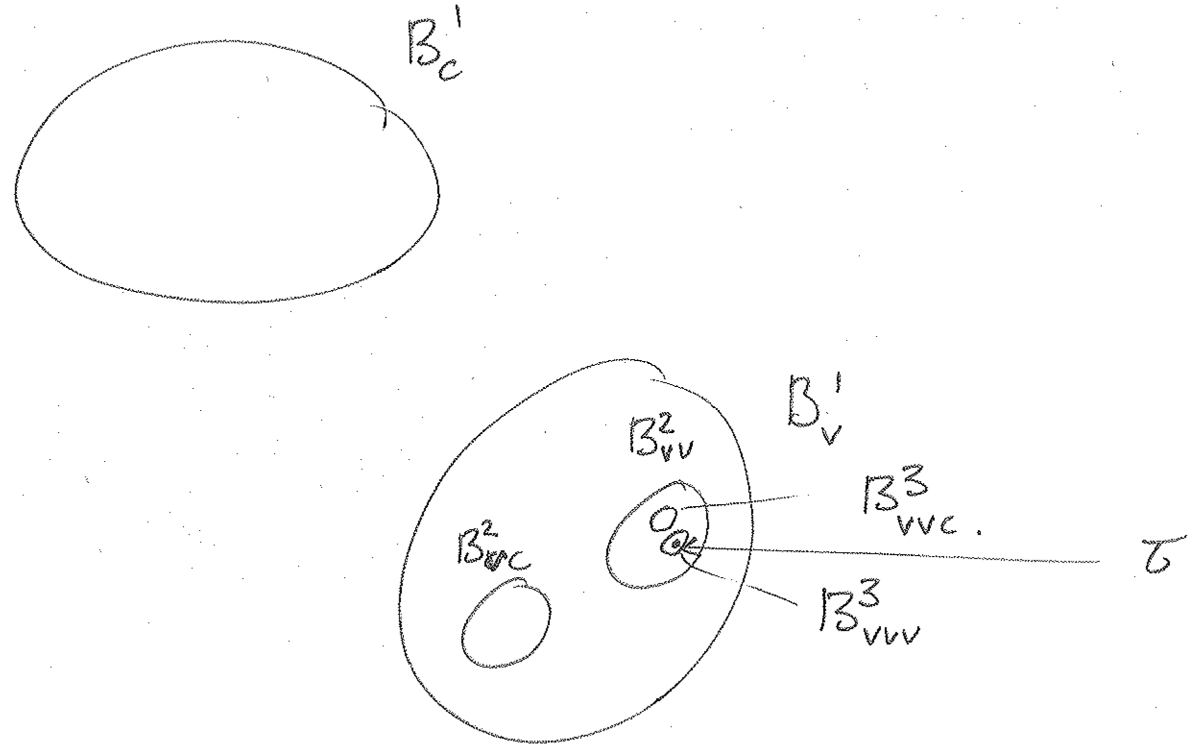
Thm ~~Zoozvan~~. There exists a analytic function $a(x)$. and $\rho < 1$ s.t.

$$R^n F\binom{x}{y} = \begin{pmatrix} f_n(x) - b_F^{2^n} a(x) y \left(1 + \mathcal{O}(\rho^n)\right) \\ x \end{pmatrix}$$

where $f_n \xrightarrow{\exp} f_*$

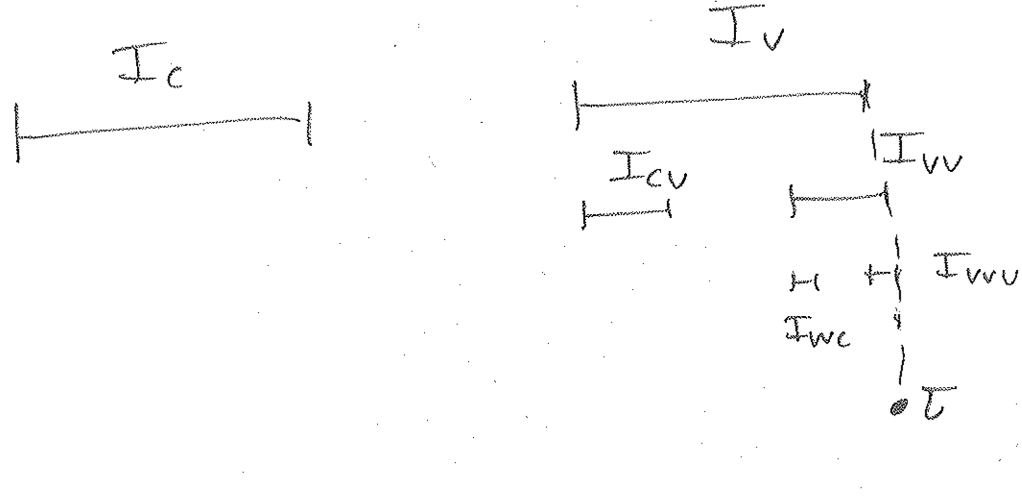
(F or-ven).

Cor: Dynamical Universality along periodic orbits:



let $\tau = \bigcap_n B^n_{\underbrace{v\cdots v}_{n}}$ be the tip.

(It plays the role of critical value of a unimodal map



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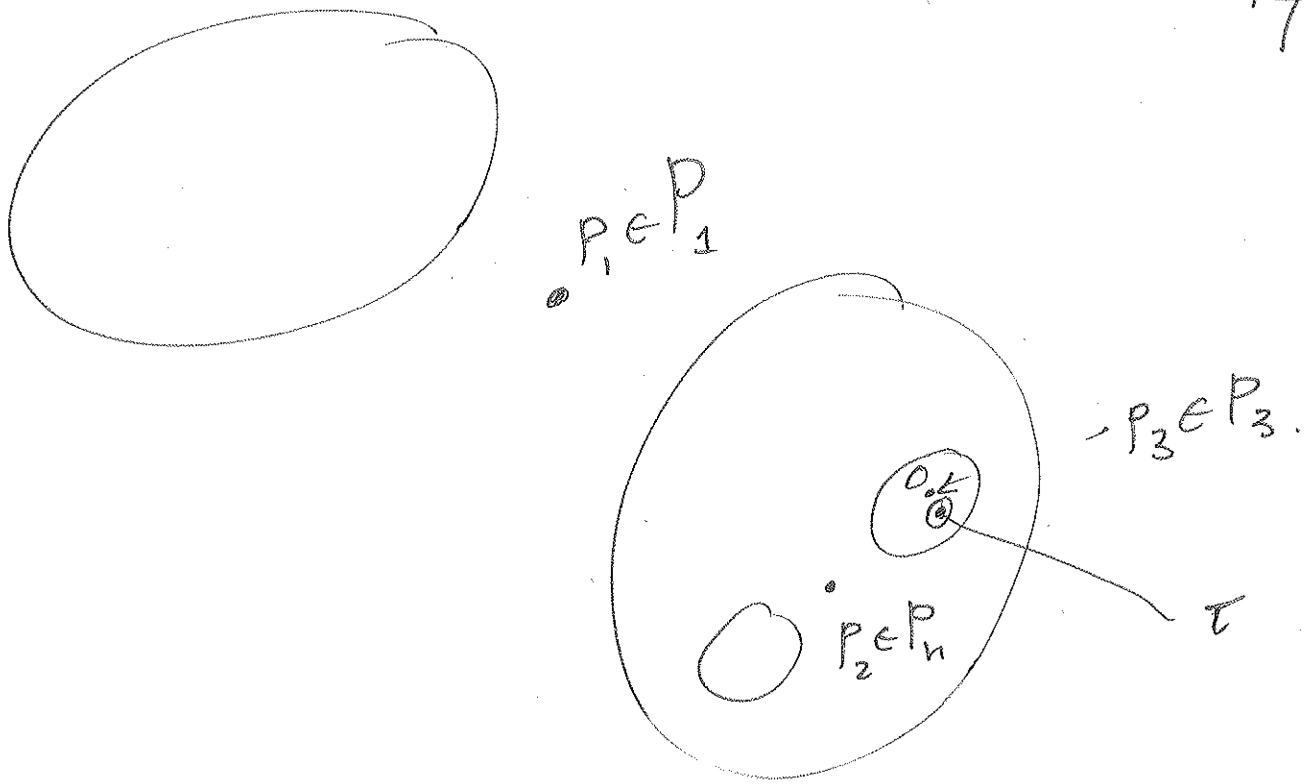
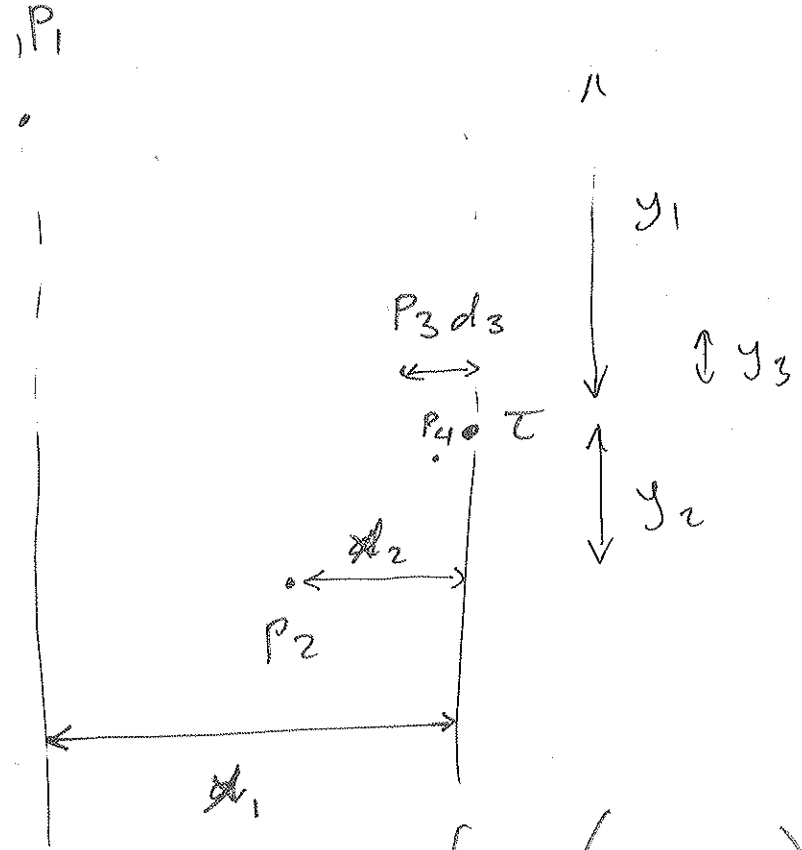




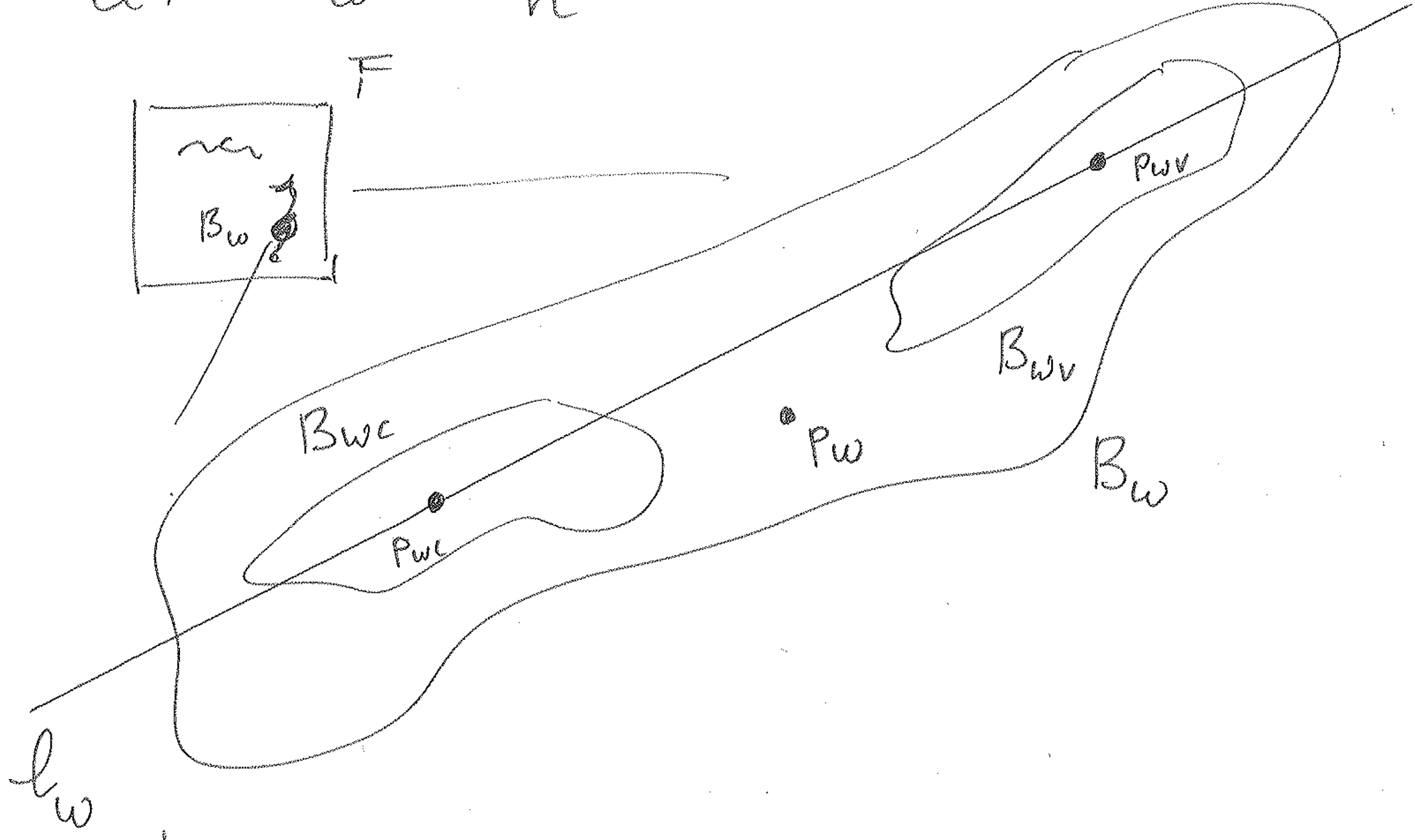
$y_n \longrightarrow 0 \qquad (2.6 \ldots)$

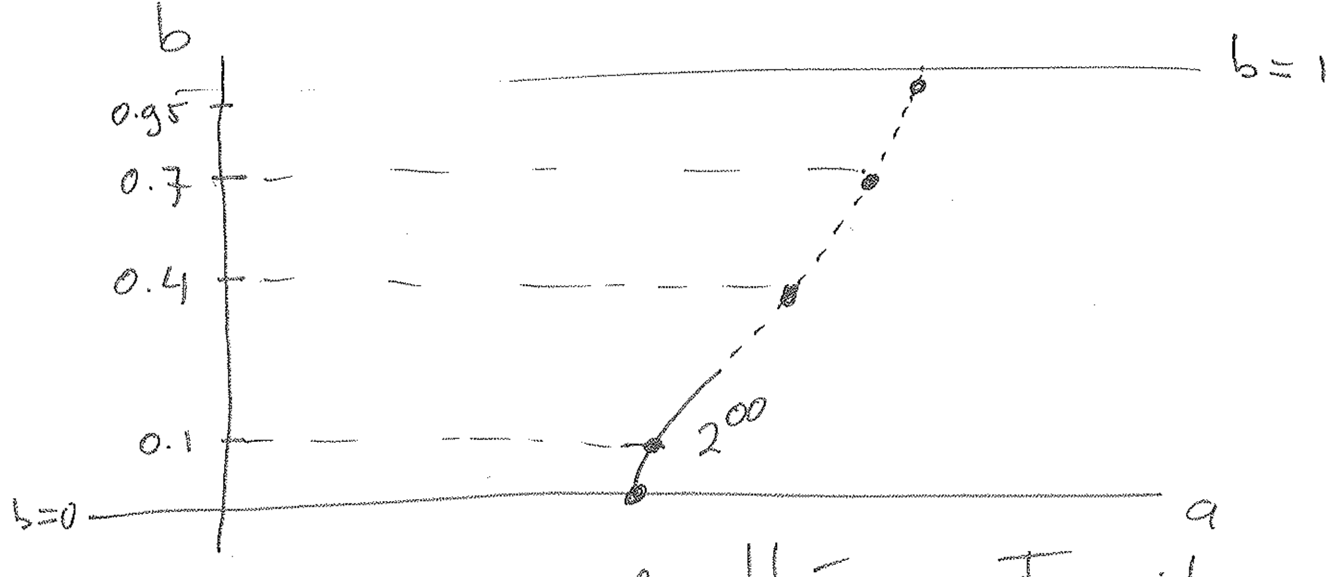
$d_n \xrightarrow{exp} 0 \qquad \left(\left(\frac{1}{2.6..}\right)^2\right).$

As in 1 Dim case ∇

Geometry Of Numerical Experiment

Let $B_\omega \in C_n$





Classical Hénon Family

The diagrams show all the lines $\ell_\omega \subset [0,1]^2$ $|\omega| \leq 12$.
For different values of b.

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The last diagram shows the distribution of the sinus of the angle with the horizontal of ℓ_ω.

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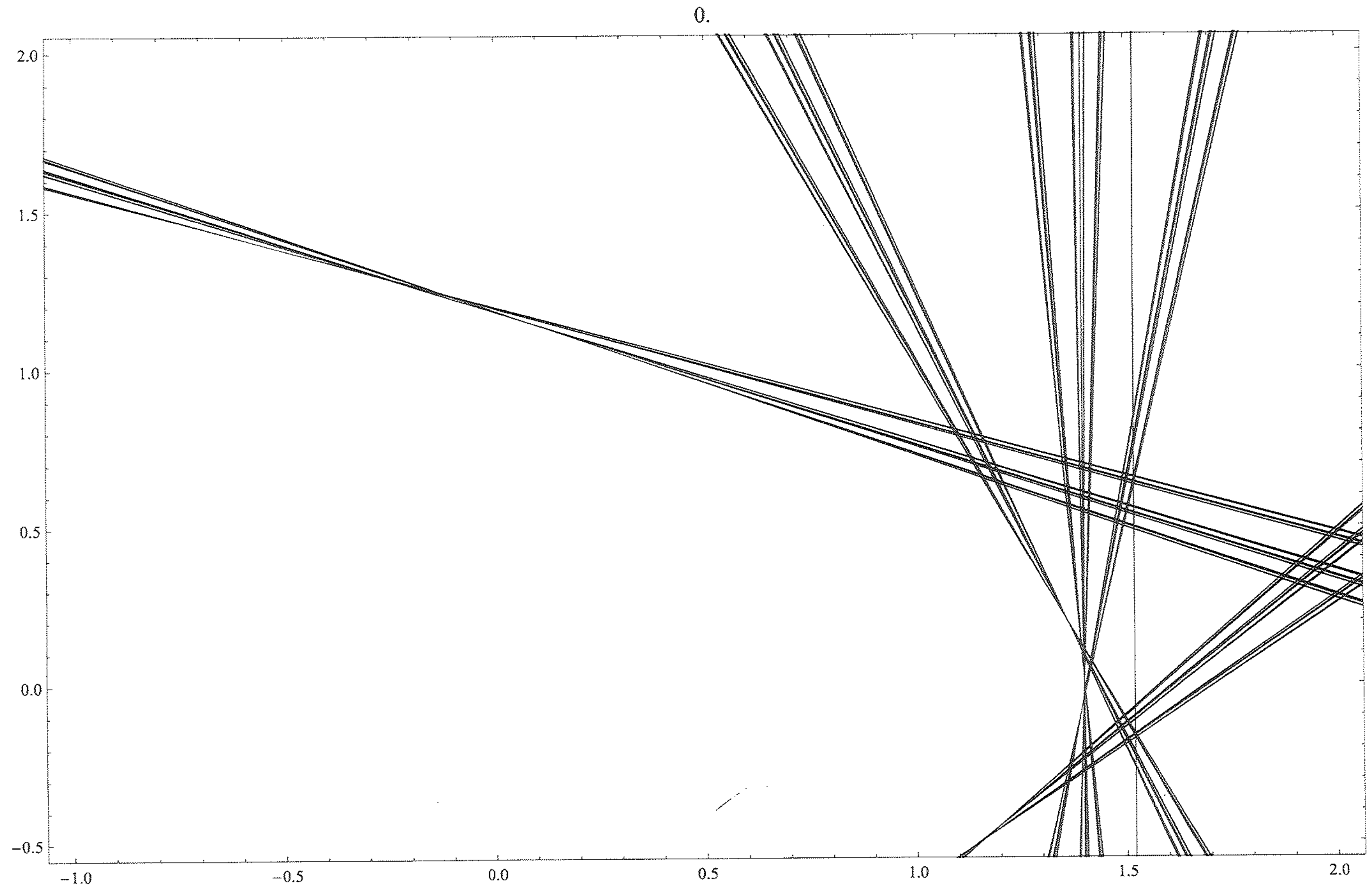
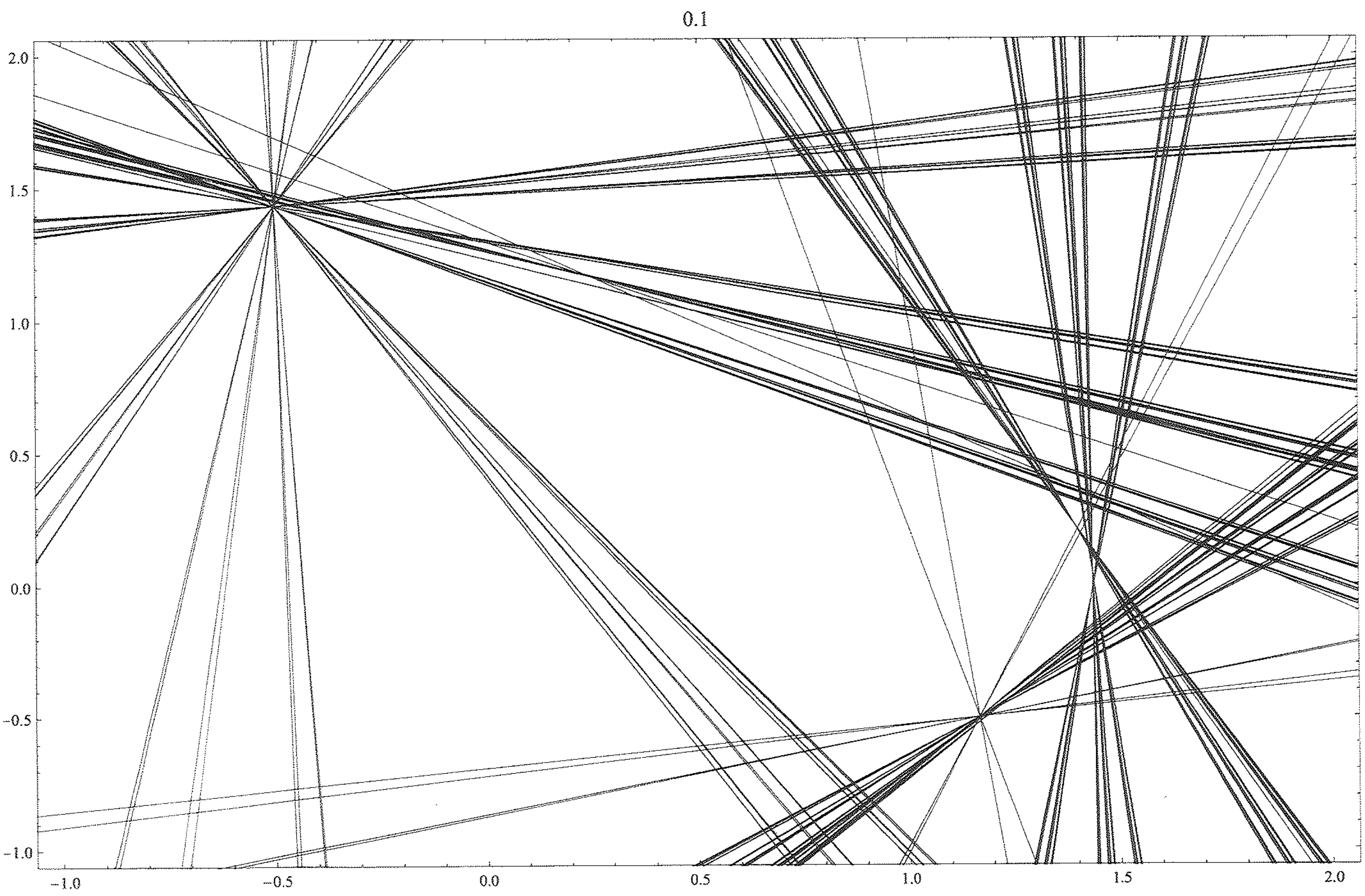
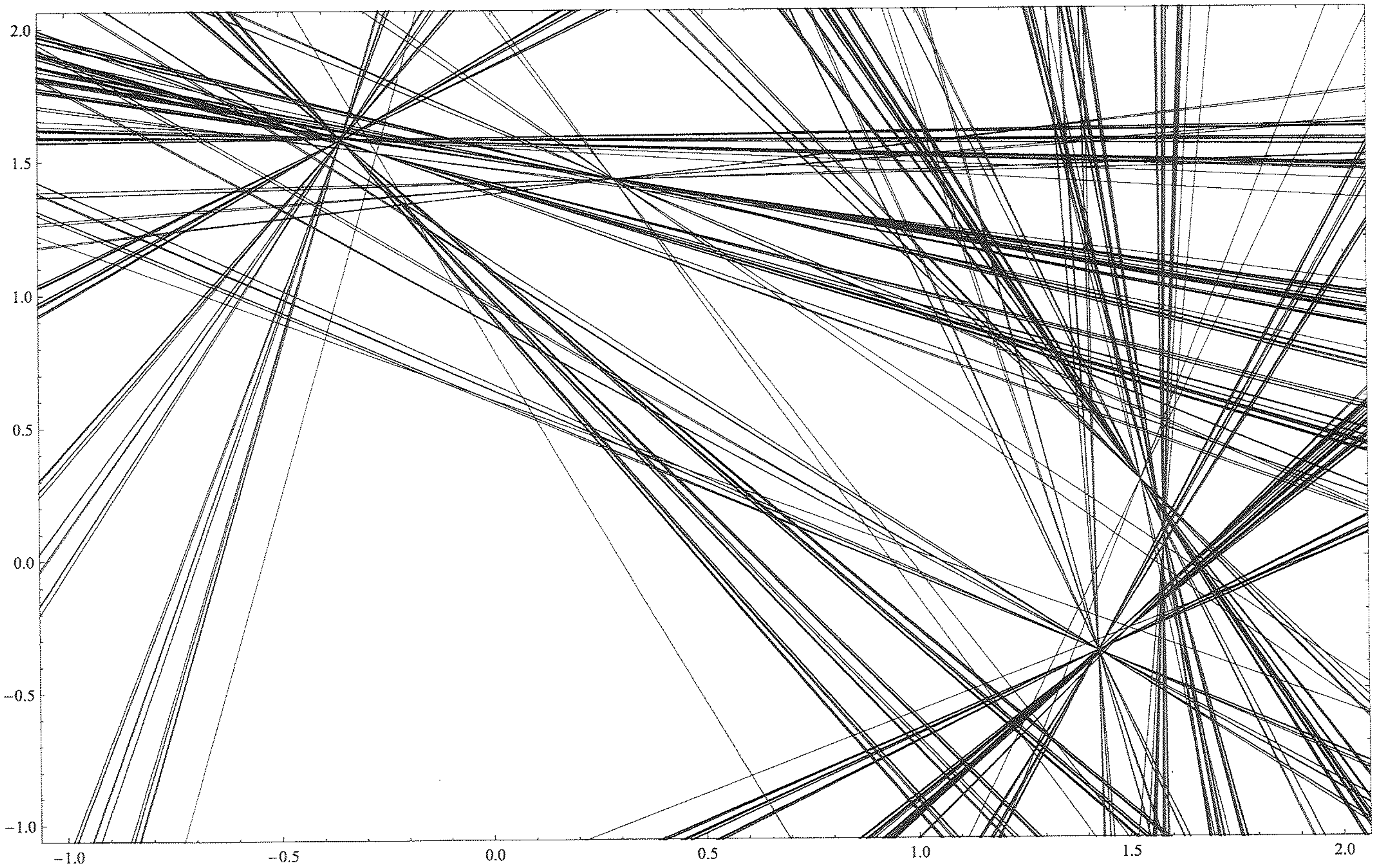
<u>Rmk</u>: the lines ℓ_ω are essentially the directions corresponding to $\lambda_1 = 1$ Lyaponov exponent.

<u>Thm</u>: $\nexists$ a continuous line field on $\mathcal{O}_F$ (wh $b_F > 0$).

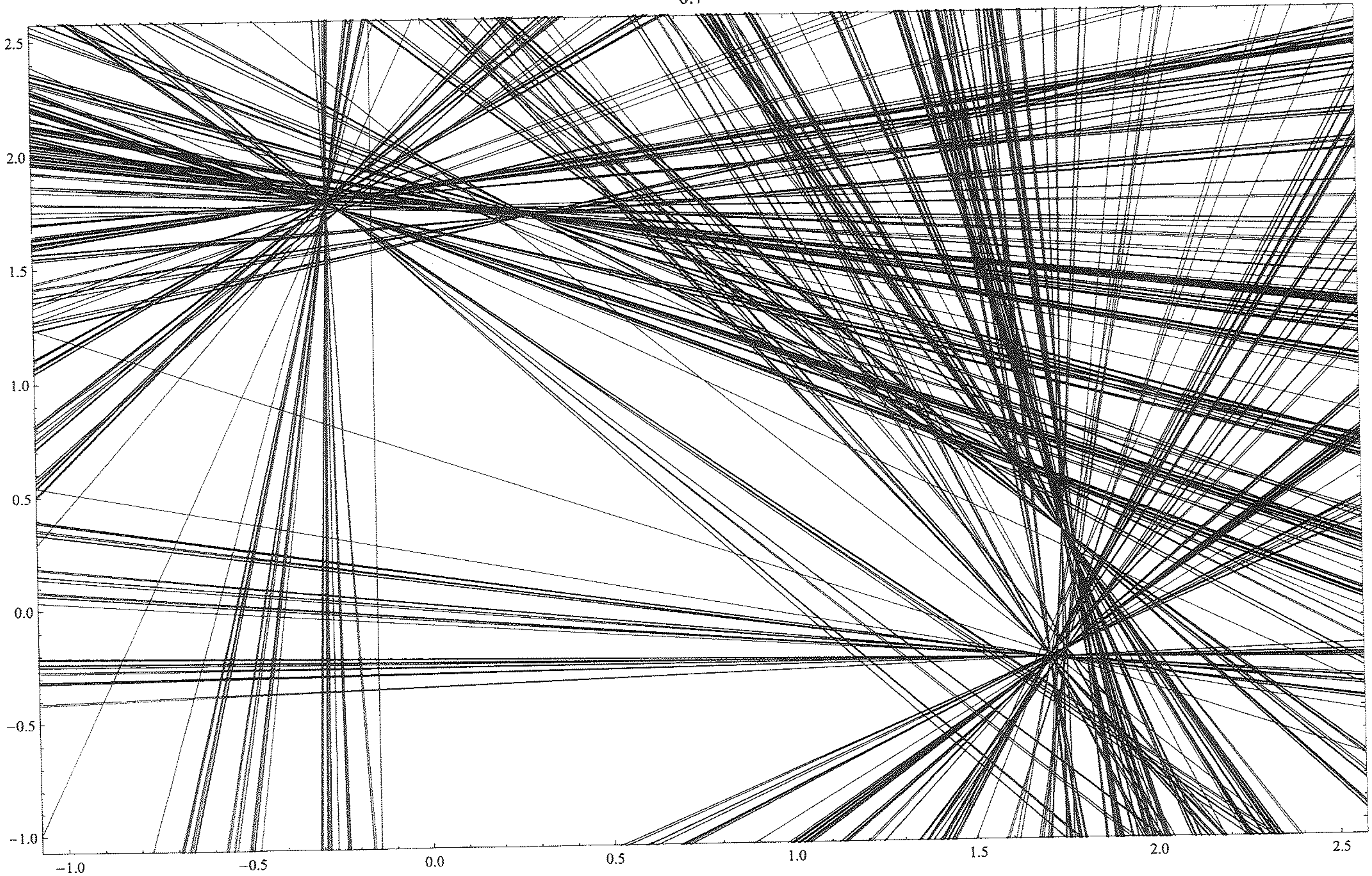

0.
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0


0.1
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.0
-0.5
0.0
0.5
1.0
1.5
2.0

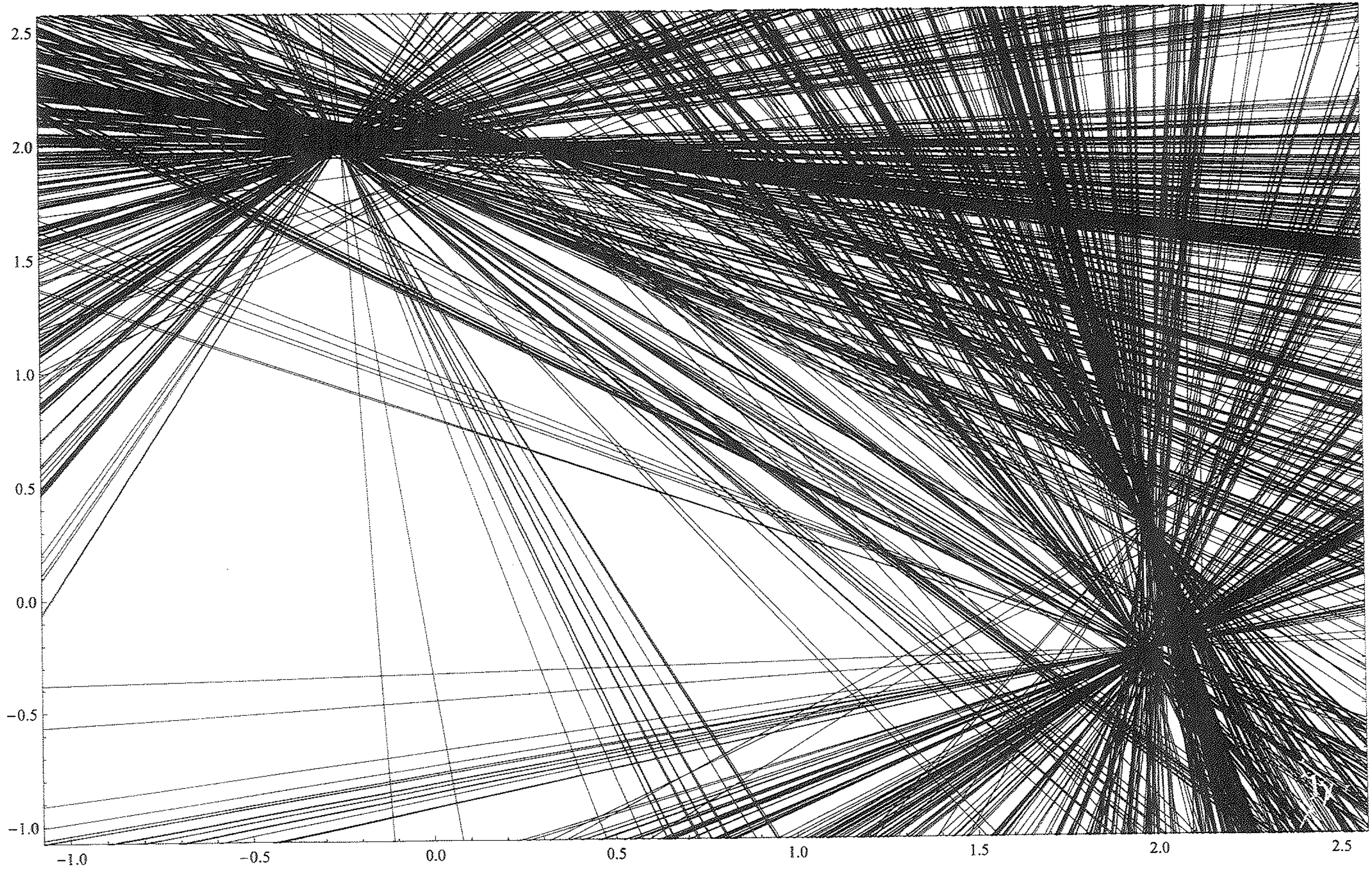
0.4



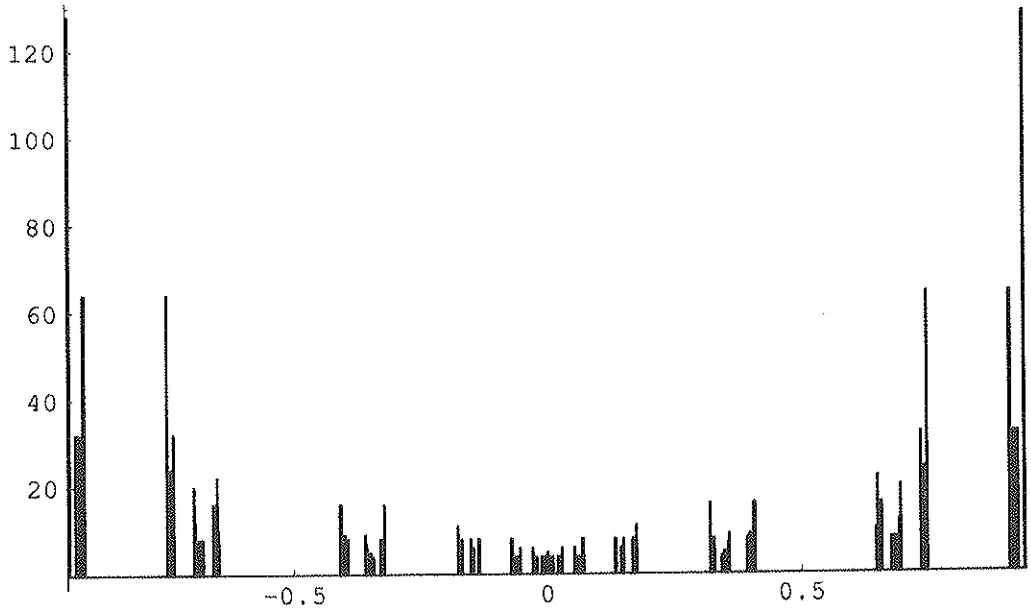
0.7



0.95



0.



0.95