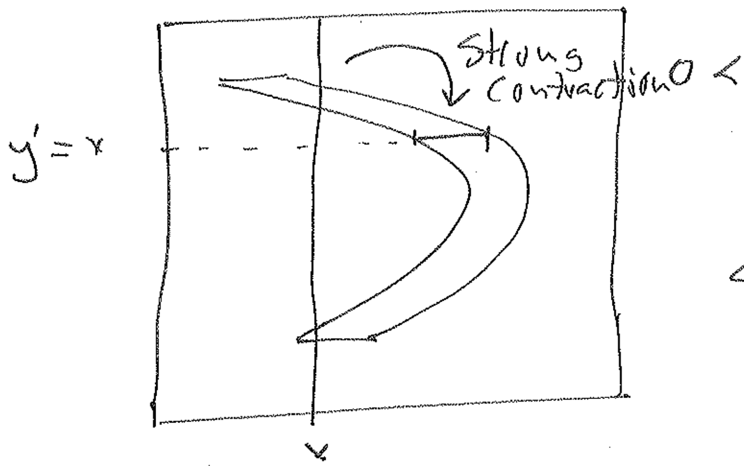
USP 2012 V

Hénon-map $F: [0,1]^2 \longrightarrow [0,1]^2$

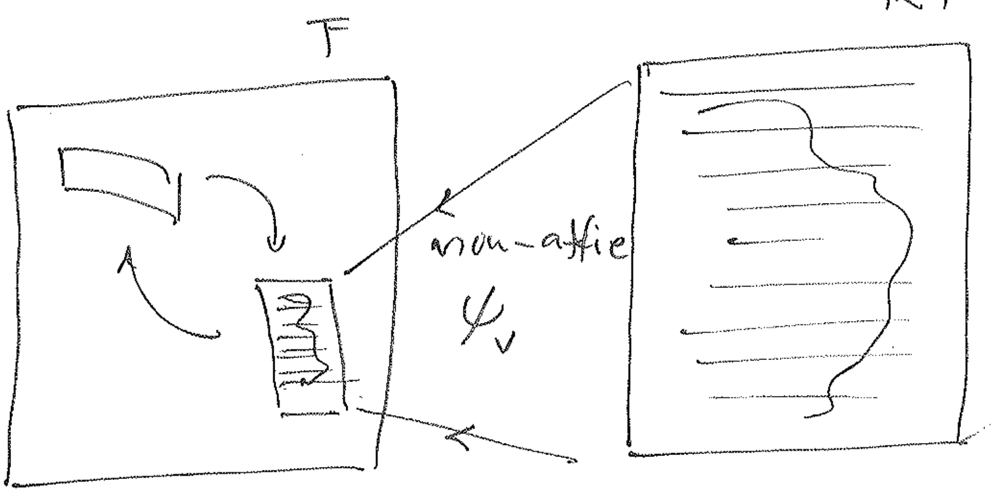
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(x) - \varepsilon(x,y) \\ x \end{pmatrix}$$



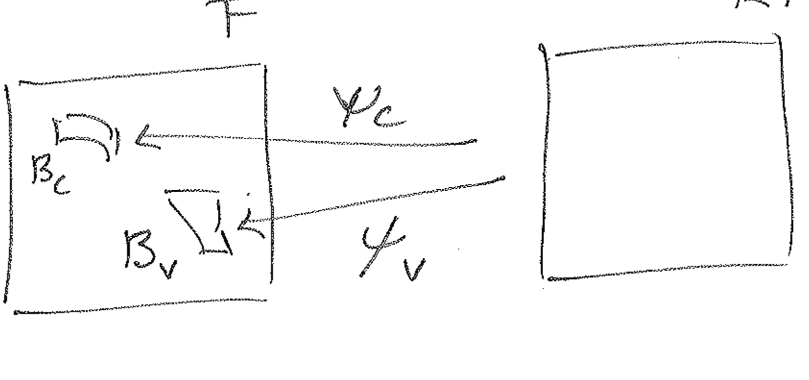
$0 < \text{Jac}\, F = \det DF = \frac{\partial \varepsilon}{\partial y} \ll 1$

Strong Dissipative.

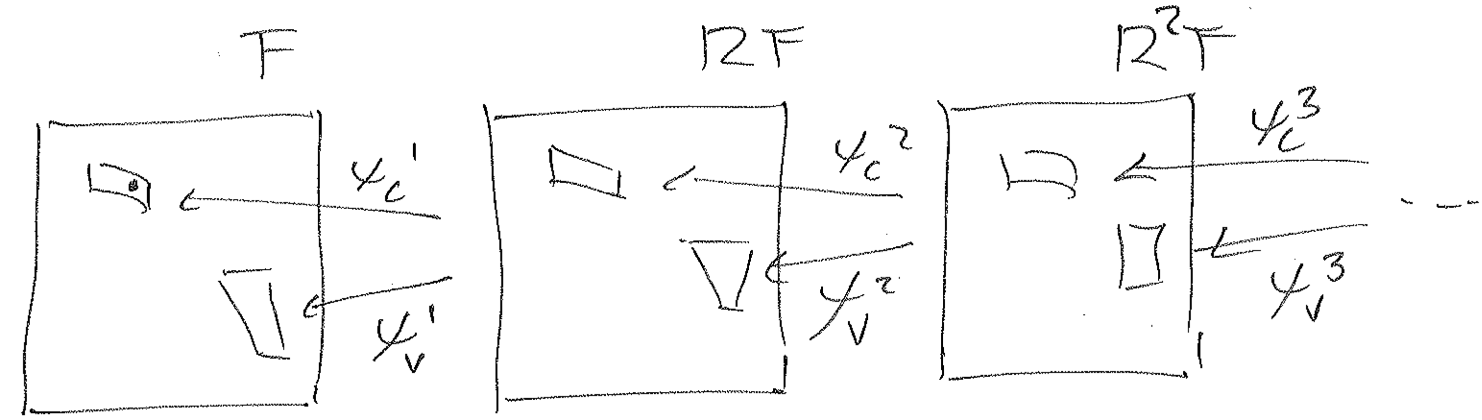
F renormalizable



$\psi_c = F \circ \psi_v$



F ∞-ren



$B_v = \text{image } \psi_v^1$

$B_c = \text{image } \psi_c^1$ $\quad \mathcal{C}_1$

$\vdots$

$$\left. B_{cv} = \text{image } \psi_c^1 \circ \psi_v^2 \right\} \; \mathcal{C}_2 .$$

$\vdots$

$$\left. B_w = \text{image } \psi_{w_1}^1 \circ \psi_{w_2}^2 \circ \cdots \circ \psi_{w_n}^n \right\} \; \mathcal{C}_n .$$

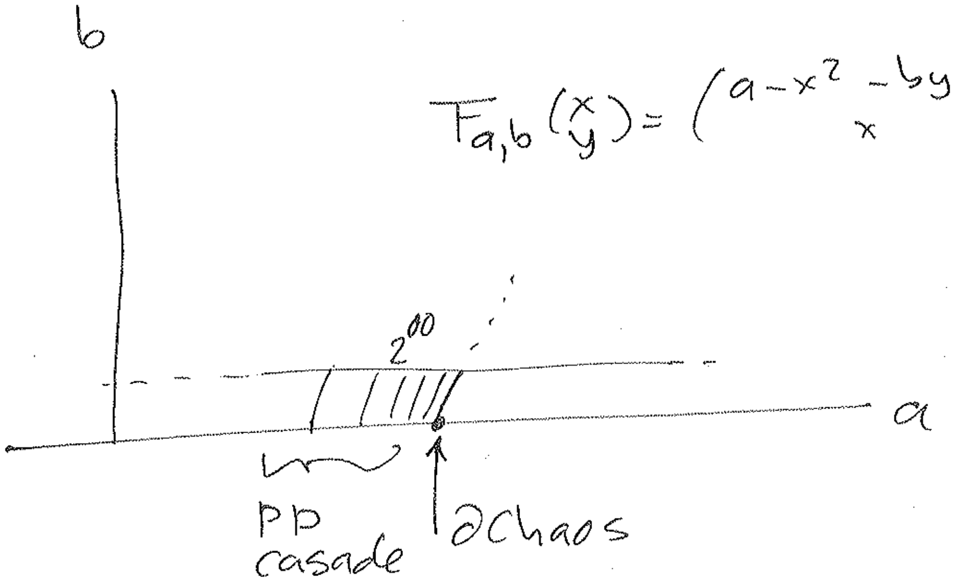
$\vdots$

$$\mathcal{O}_F = \bigcap \mathcal{C}_n . \quad \text{Cantor attractor}$$

$$b_F = e^{\int \ln \operatorname{Jac} F \, d\nu} .$$

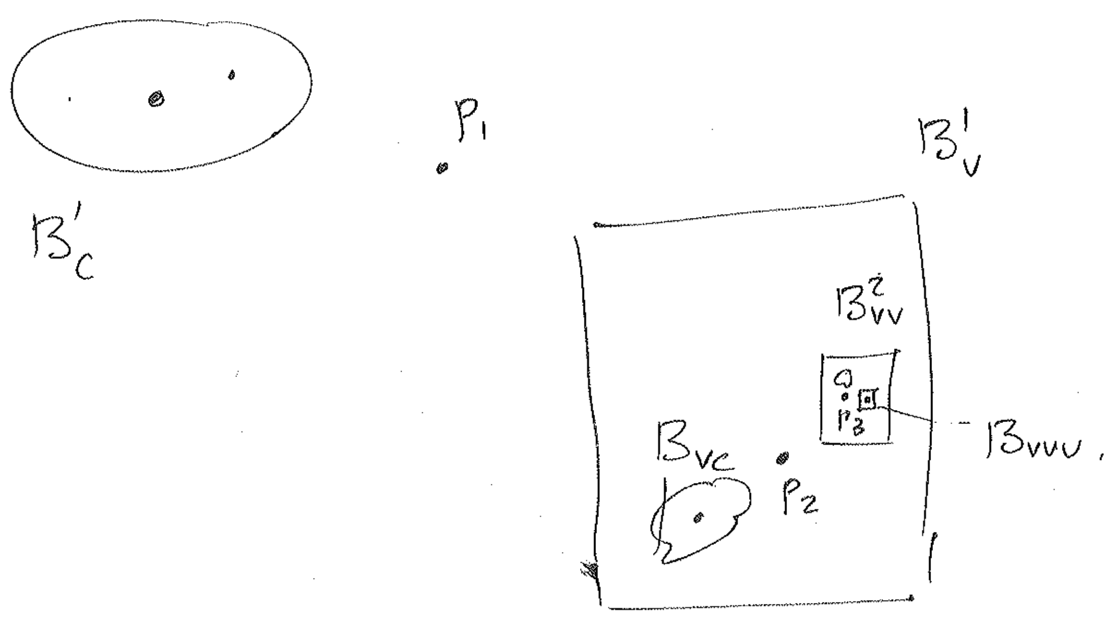
Rmk

$$F_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - x^2 - by \\ x \end{pmatrix}$$



Parameter Universality.

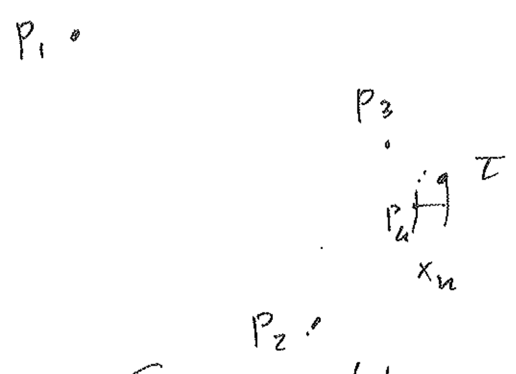
Rmk:



Let τ tip

$$\tau = \bigcap B_{\underbrace{v \dots v}_{n}}$$

$X_n \longrightarrow 0$



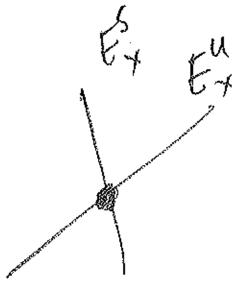
$(2.66.)^2$

~~Def~~ Phase Space Universality

Bad News

<u>Thm</u>: there is no continuous invariant line field on $\mathcal{O}_F$. ($b_F > 0$).

<u>Thm</u> $\lambda_1 = 1 \quad \lambda_2 = b_F$ (Lyapunov exponents).



and E^1_x, E^2_x are <u>not</u> continuous (only measurable).

<u>Thm</u> $\tilde{F}, F$ ∞-ren. with

$$b_{\tilde{F}} < b_F$$

$h: \mathcal{O}_{\tilde{F}} \longrightarrow \mathcal{O}_F$ is at most

$$C^{\underbrace{\frac{1}{2}\left(1+\frac{\ln b}{\ln \tilde{b}}\right)}_{<1}}.$$

<u>Example</u>:

$h: \mathcal{O}_{f_*} \longrightarrow \mathcal{O}_F$ is at most $C^{1/2}$.

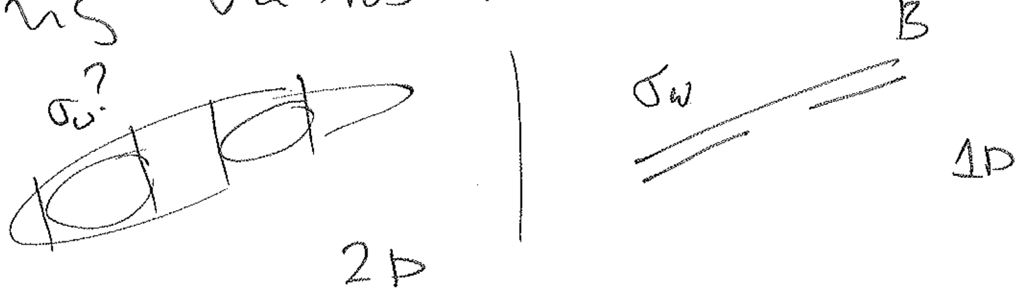
No Rigidity

Conclusion:

Some universal geometrical properties of $\mathcal{O}_{f_*}$ (1D) survive whe $b_F > 0$ ($p_n \to \tau\ (2.66)^2$.)

But $\mathcal{O}_F$ has definitely a different geometry th $\mathcal{O}_{f_*}$ (1D).

Rmk: At this point its not clear whether we can still speak of scaling ratios?



Rmk: Even is the pieces B_w are still essentially little ~~line~~ fattened line segments, the directions are highly not continuous, see experiment and first Thm.

<u>Thm</u>$^{(HLM)}$: $\exists$ $A \subset [0,1]$ dense G_δ of full measure.

<u>If</u> $b_F \in A$ <u>then</u> $\mathcal{O}_F$ has unbounded geometry.

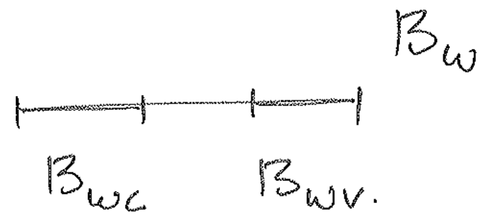
$$A : \left\{ b^{2^n} \asymp \sigma^k \;\middle|\; \text{infinitely many } n, k \right\}.$$

$$(\sigma = 2.66..).$$

Geometrical bounds in 1D

$\forall w \qquad 0.1 < \sigma_w < \cancel{\ } \, 0.9$

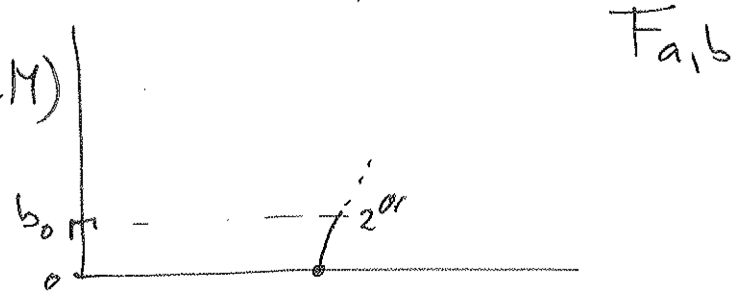
So



~~intervals are~~

The intervals B_{wc}, B_{wv}, B_w are of comparable size. ($\forall w$). (A priori bounds)

<u>Thm</u> (HLM)

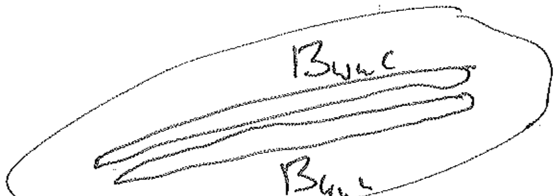


$\exists\, b_0 > 0$ a.e. $b \in (0, b_0]$ σ_F

<u>has no a priori bounds:</u>

$\exists\, B_{w_n} \subset C_n$ s.t.

B_{w_n} with.



$\cancel{\mathrm{Dist}(B_{w_n c}, B_{w_n v})}$

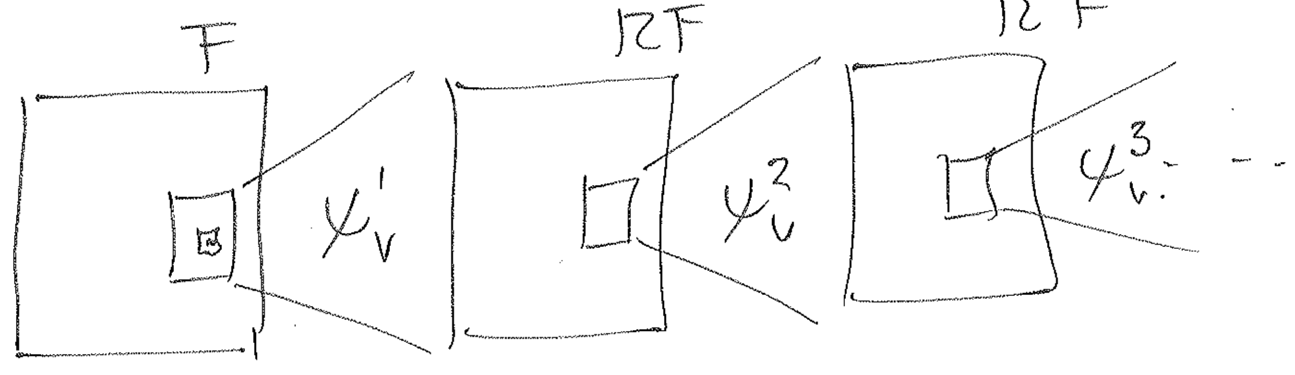
$$\frac{\mathrm{dist}(B_{\omega_n c}, B_{\omega_n v})}{\mathrm{diam}(B_{\omega_n})} \longrightarrow 0$$

~~the cause of the~~

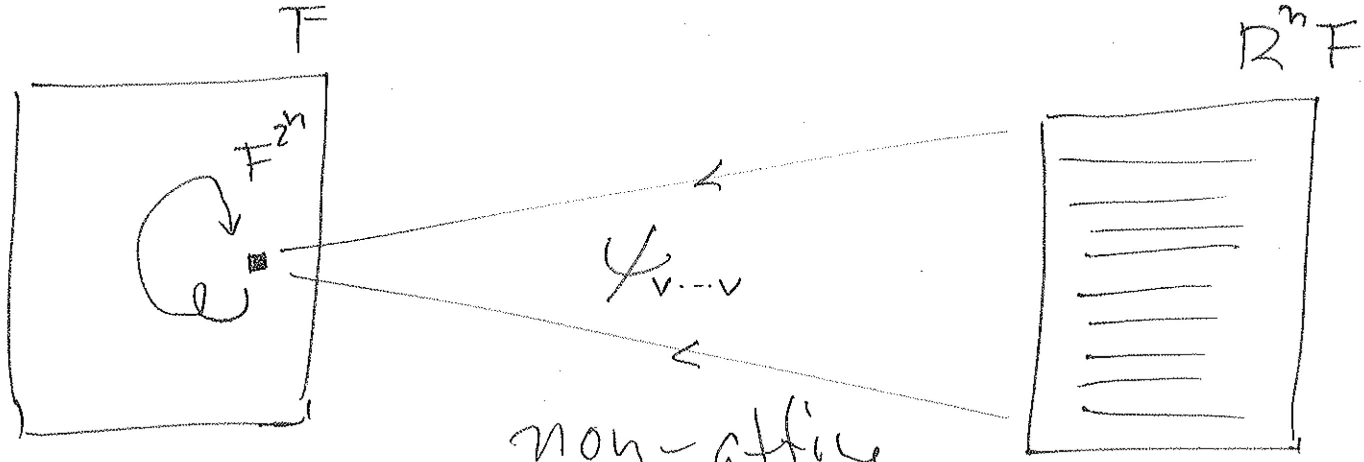
The (initial) cause of the destruction of 1D geometry is

Tilt

let F ∞-ren



$$\psi_{\underbrace{v\cdots v}_{n}} : [0,1]^2 \longrightarrow B_{\underbrace{v\cdots v}}$$



non-affine
but horizontal.

$$R^n F \sim \begin{pmatrix} f_*(x) \\ x \end{pmatrix}$$

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To understand the geometry in $B_{v\cdots v}$ we need to study $\psi_{v\cdots v}$,
We do know the geometry of $R^n F$.

<u>Thm</u>

$$\psi_{v\cdots v} = \begin{pmatrix} 1 & t_F \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \sigma^{2n} & 0 \\ 0 & (-\sigma)^n \end{pmatrix} \circ \begin{pmatrix} S_n(x,y) \\ y \end{pmatrix}$$

where.

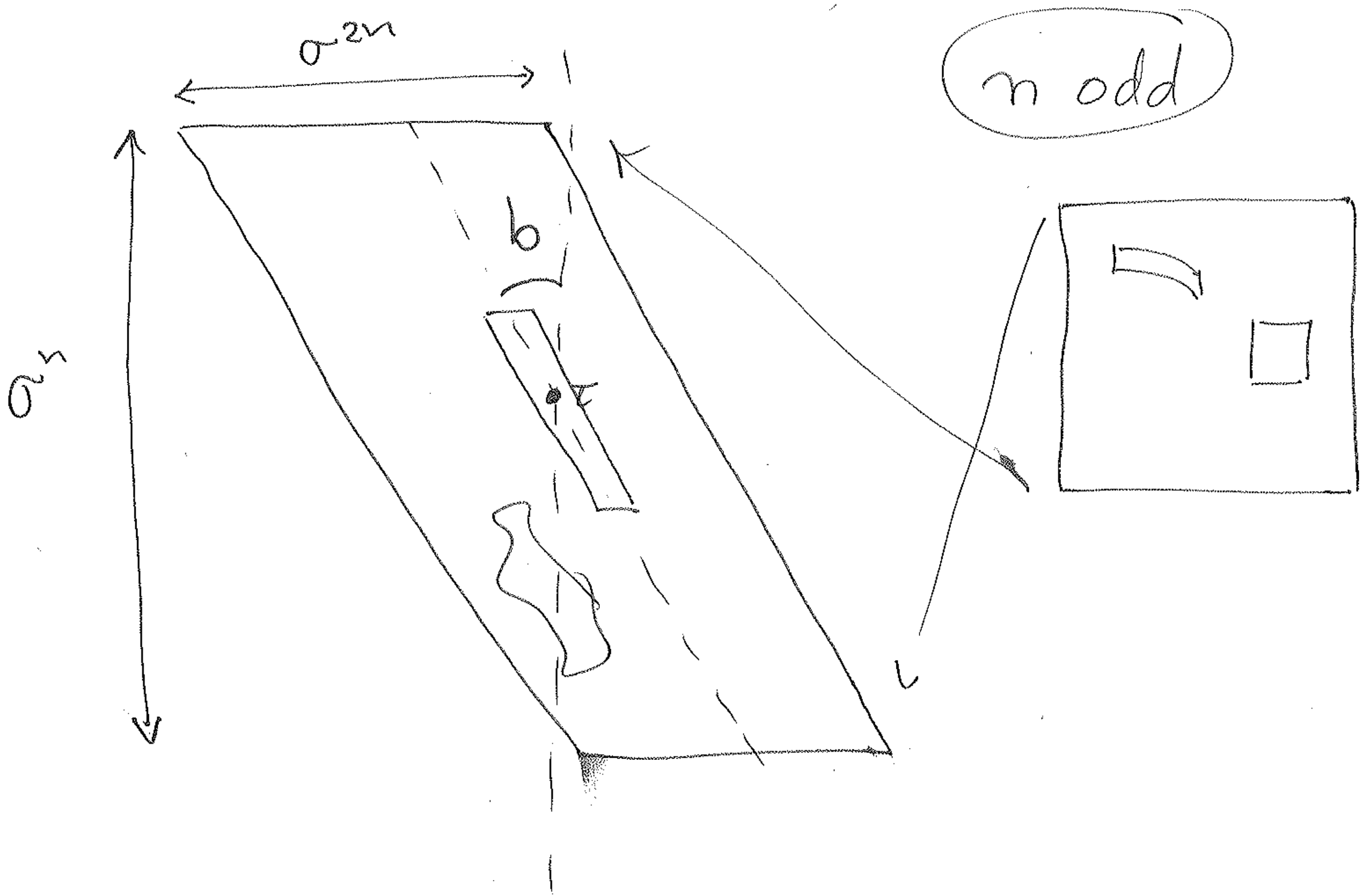
$t_F \asymp b_F$

$\sigma = 2.66..$

$S_n(x,y) = v(x) + a_F y^2 + O(\rho^n)$.

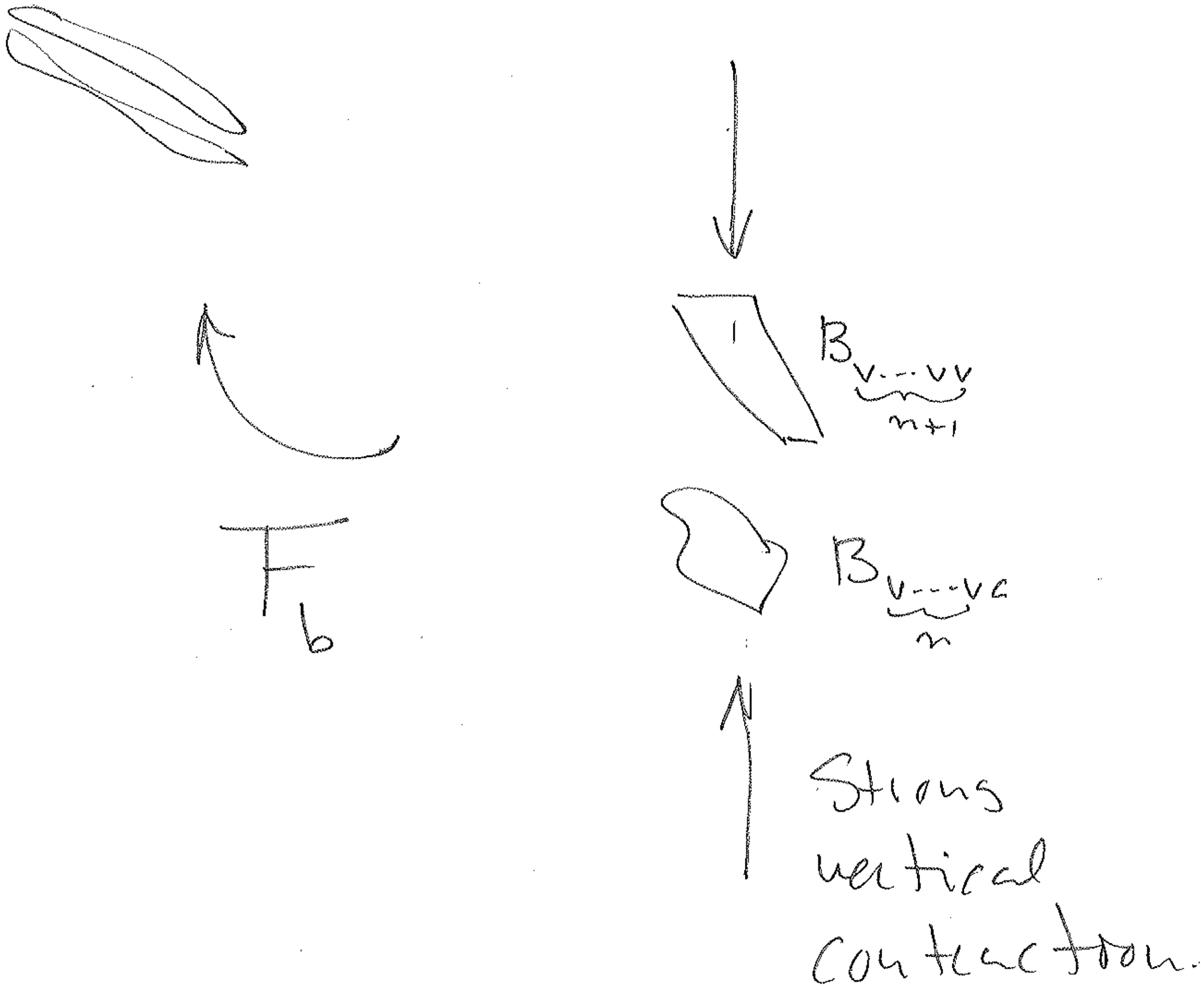
$v(x)$ analytic function.

n odd



$B_{\underbrace{v\cdots v}_{n}}$

$$\boxed{b\sigma^n \asymp \sigma^{2n}}$$



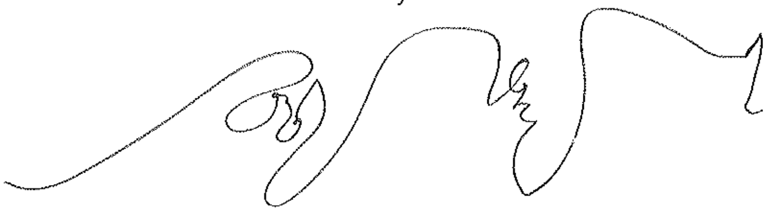
$b_F > 0$ 2D Dynamics

$\neq$

1Dim Dynamics.

Good News

⊛ O_F is not part of a smooth curve



However,

Thm O_F, $b_F > 0$, is part of a rectifiable curve (a curve with finite length).

⊛ Scaling Structure

What happened with Σ_{f_*}?

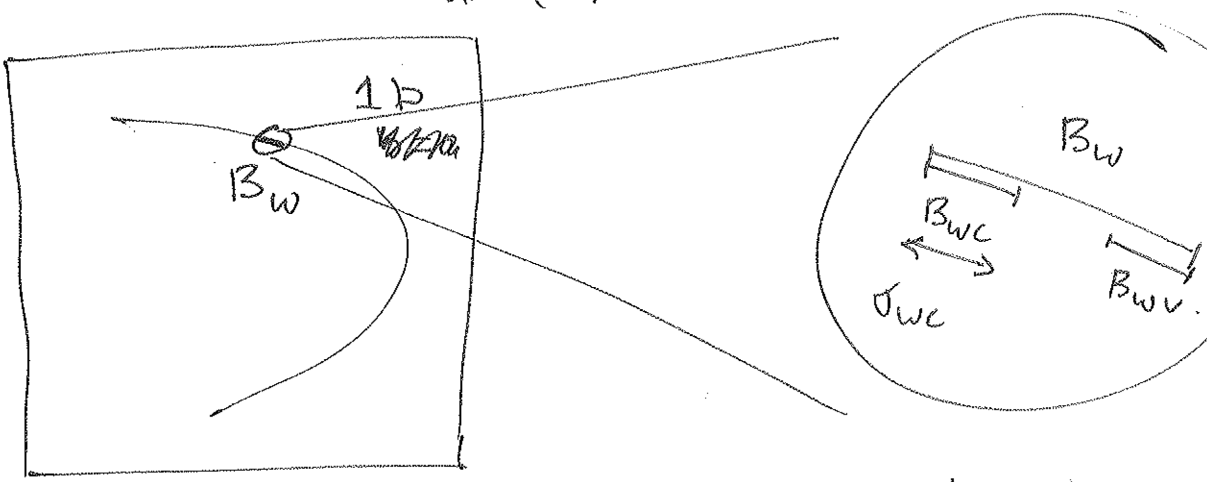
Is there still something left of the 1D scaling structure?

Feigenbaum experiment: yes.

Geometry of σ_F

$$F_* = \begin{pmatrix} f_*(x) \\ x \end{pmatrix}$$

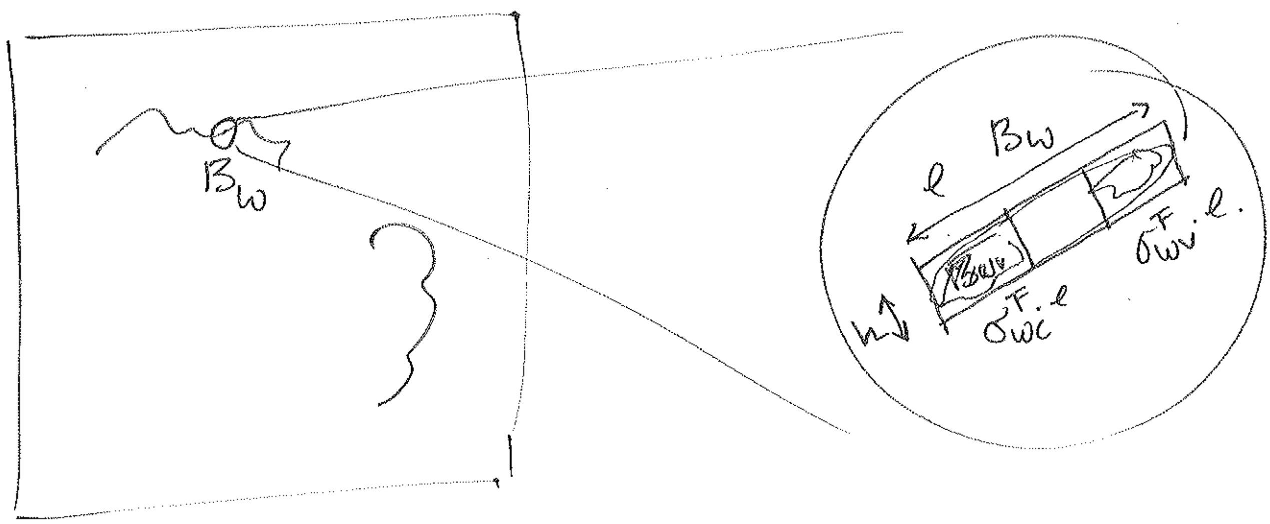
<u>1D</u>



$$\frac{|B_{wc}|}{|B_w|} = \sigma_{wc} \in \Sigma_{f_*}$$

$b_F > 0$ F

2D



B is ε-standard:

- $|\cancel{\ }h| \leq \varepsilon \cdot \ell$
- $|\sigma_{w_\cdot}^{F} - \sigma_{v_\cdot}| \leq \varepsilon$.

$\mathcal{S}_n(\varepsilon) = \{\varepsilon\text{-standard pieces of } \mathcal{C}_n\}$.

<u>Thm</u> (Probabilistic Universality)

$\exists\, \theta < 1$

$$\nu(\mathcal{S}_n(\theta^n)) > 1 - \theta^n$$

=

$\mathcal{C}_n$ has 2^n pieces and at most $\theta^n \cdot 2^n$ of them are not as in ~~B~~ 1D.

=

$f, g \in$ ~~[illegible]~~ ∂Chaos (1D)

$h : \mathcal{O}_f \to \mathcal{O}_f$ $C^{1+\alpha}$

(Rigidity).

<u>Thm</u> (Probabilistic Rigidity).

$\exists\, \beta > 0$

$X_1^F \subset X_2^F \subset \cdots \quad \subset \mathcal{O}_F$.

with

- $\mu(X_n^F) \longrightarrow 1$
- X_n^F contained in $C^{1+\beta}$ curve.
- $h : \mathcal{O}_F \longrightarrow \mathcal{O}_{f_*}$ is $C^{1+\beta}$ on X_n^F

$h : X_n^F \longrightarrow h(X_n^F) \subset \mathcal{O}_{f_*}$ $C^{1+\beta}$.

<u>Thm</u>: If $b_{F_1} = b_{F_2}$ then

$h : \mathcal{O}_{F_1} \to \mathcal{O}_{F_2}$ preserves.

$h : X_n^{F_1} \longrightarrow X_n^{F_2}$ and is $C^{1+\beta}$.

Thm

$$HD_\nu(\mathcal{O}_F) = HD_\nu(f_*).$$

—— // ——

Conclusion:

Universality and Rigidity has a probabilistic nature in higher dimensions.