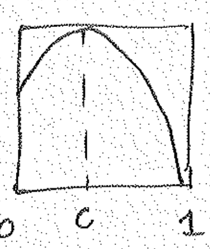
USP 2012 III

1D (Period Doubling) Renormalization &

$\mathcal{U} = \{ f : [0,1] \to [0,1] \mid$

- 

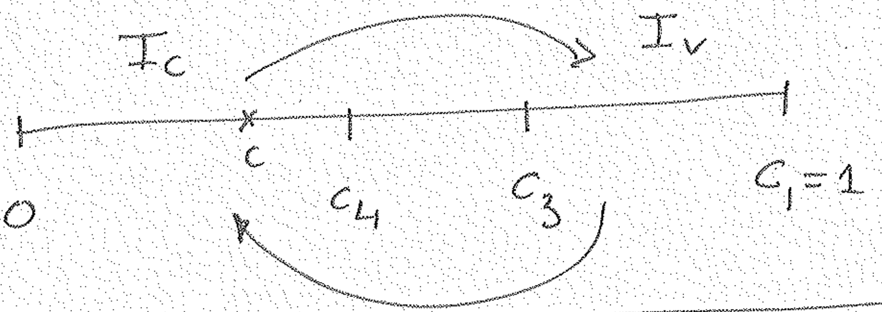
 C^3

- $Df(c) = 0 \quad D^2 f(c) < 0$
- $f(c) = 1 \quad f(1) = 0 \}$.

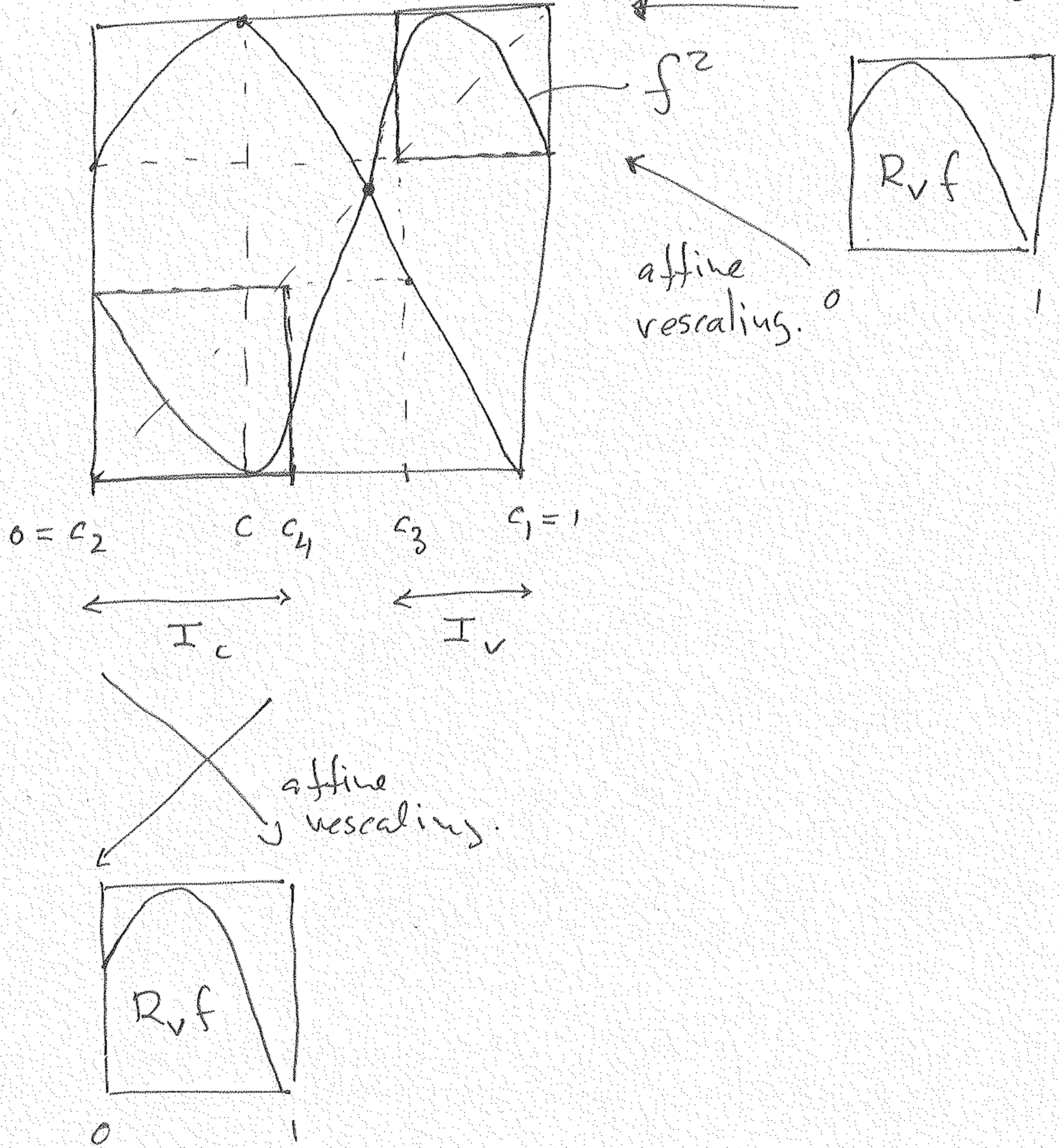
Let $c_n = f^n(c) \quad n \geq 0$

__Def__ f is renormalizable if

$$0 = c_2 < c < c_4 < c_3 < c_1 = 1$$



__Rmk:__ $\boxed{f \text{ exchanges } I_c \text{ and } I_v.}$



Precise definition $R_v f$: ($R_c f$ similar)

Let $h : [0,1] \longrightarrow [c_3, 1]$

$$h(x) = c_3 + (c_1 - c_3) \cdot x$$

$$R_v f(x) = h^{-1} \circ f^2 \circ h(x)$$

$\mathcal{U}_0 \subset \mathcal{U}$ subset of renormalizable maps.

<u>Rmk</u>: Renormalizability is a topological property.

$$R_{c,v} : \mathcal{U}_0 \longrightarrow \mathcal{U}.$$

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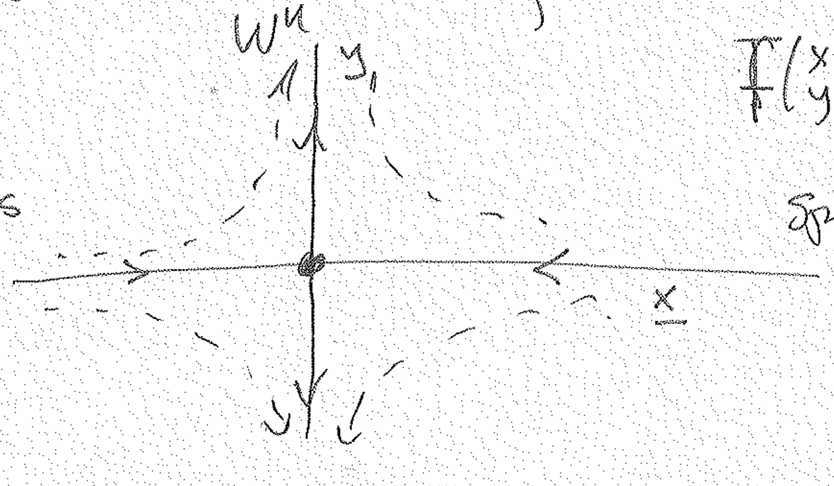
Recall the notion of hyperbolicity.

~~The simplest hyperbolic system~~

Hyperbolic systems are supposed to be the simplest and best understood systems. Among them the simplest of all is a saddle point.



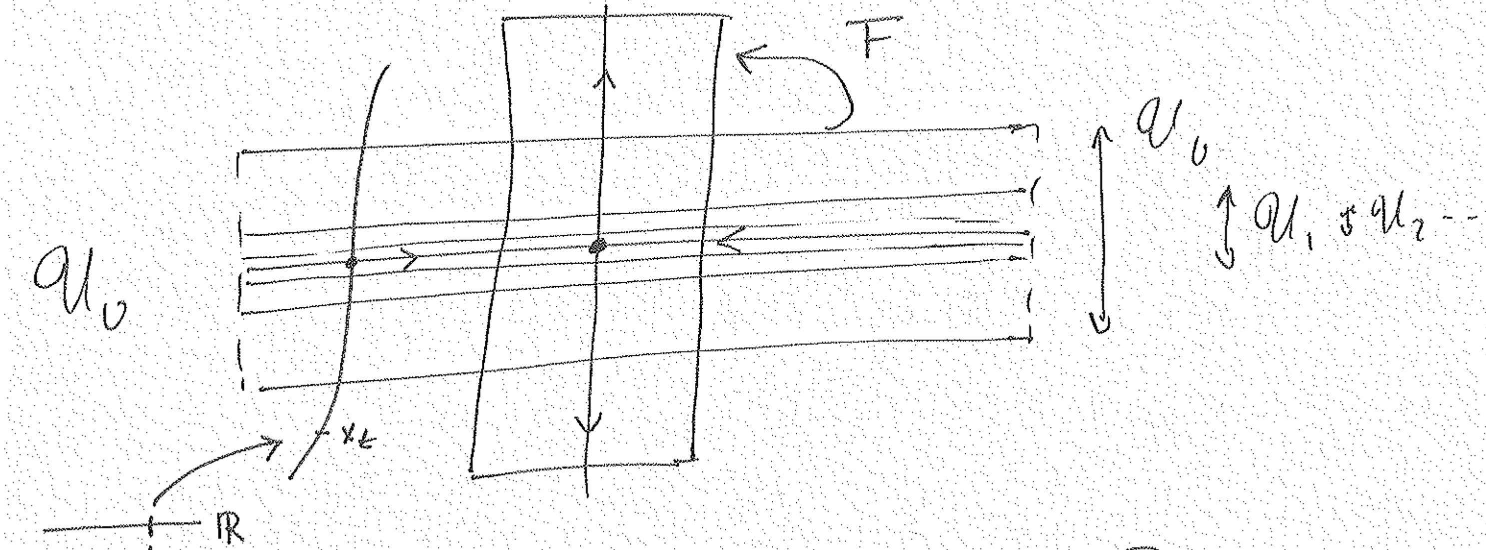
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \Phi \underline{x} & 0 \\ 0 & \delta y \end{pmatrix}$$

$\operatorname{Spec} |D\Phi| < 1$

$\delta > 1$

Suppose F is only defined on

$$\mathcal{U}_0 = \{ |y| \leq 1 \}$$



$$\mathcal{U}_n = \{ (x,y) \in \mathcal{U}_0 \mid F^n_{(x,y)} \text{ is defined} \}.$$

$$\mathcal{U}_n = \{ |y| \leq \delta^{-n} \}$$

Consider a family

$\mathbb{R} \ni t \mapsto (x_t, y_t) \in \mathcal{U}$, with $y_0 = 0$

Let $U_n = \{ t \in \mathbb{R} \mid (x_t, y_t) \in \mathcal{U}_n \} \Longrightarrow$

$$U_n = [a_n, b_n]$$

$$\cancel{a_n \leq \delta^{-n}} \quad \cancel{b_n \leq \delta^{-n}}$$

(*) $a_n / a_{n-1} \longrightarrow \delta \qquad b_n / b_{n-1} \longrightarrow \delta$

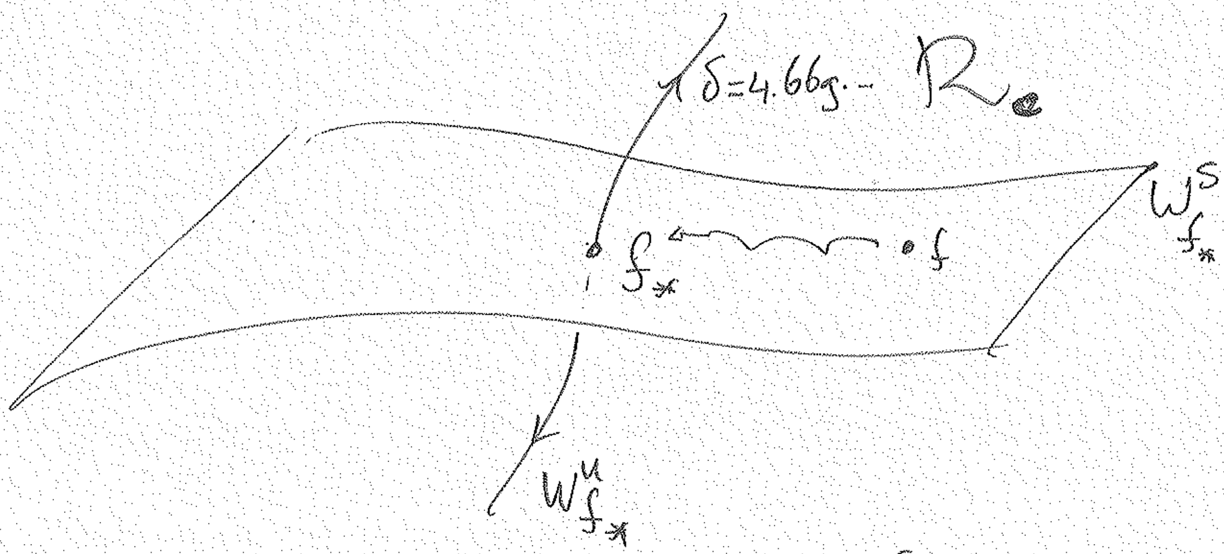
And

$$\circledast \quad F^n((x_0,y_0)) \xrightarrow{\exp} 0$$

<u>Conjecture</u> (Coullet & Tresser).

$$R_{c}: \mathcal{U}_0 \longrightarrow \mathcal{U}$$

is hyperbolic. It has a unique saddle fixed point with 1D unstable direction ($\delta = 4.669\ldots$) and a codim 1 stable



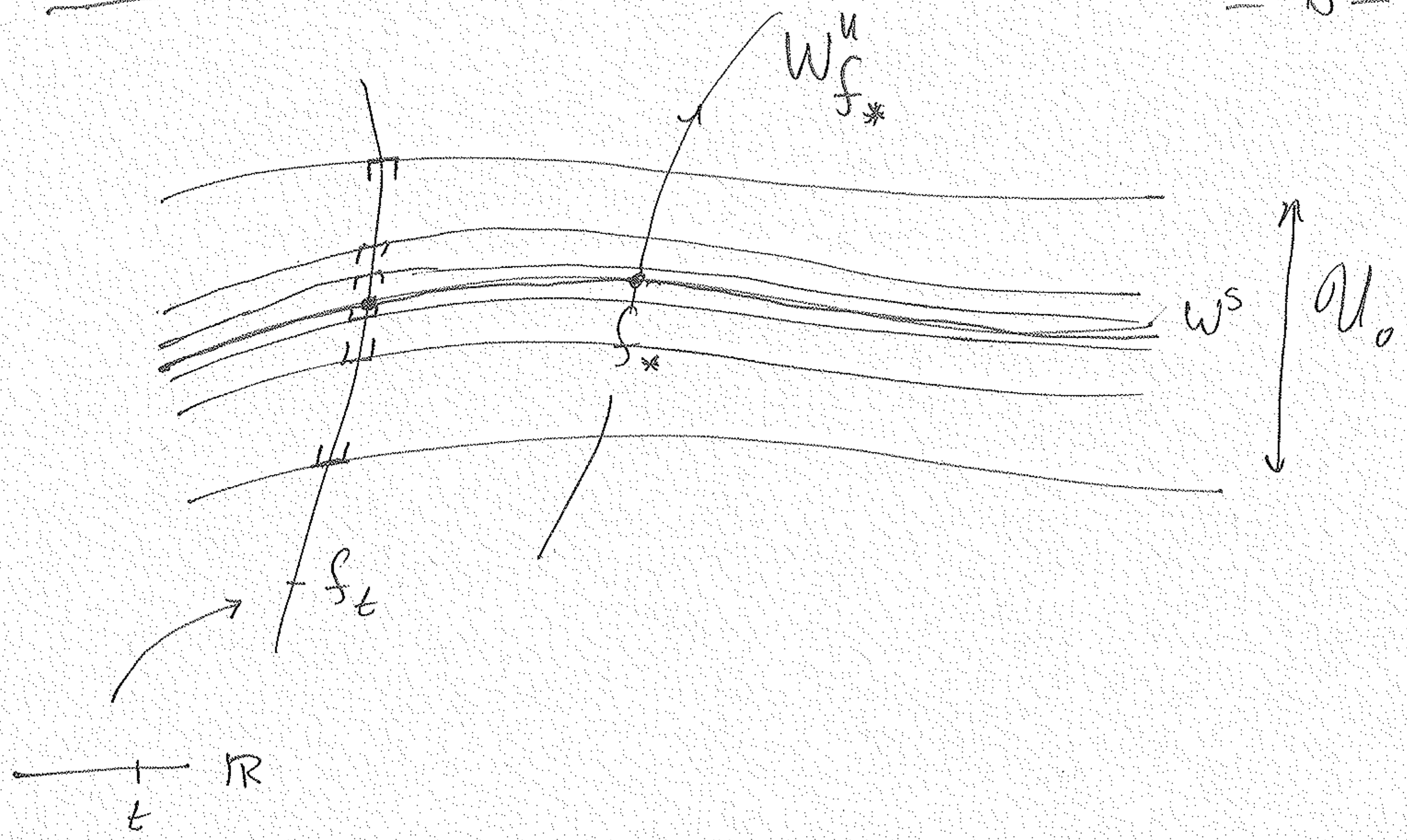
$$\{\infty\text{-ren}\} = \partial\,\text{Chaos} = W^s_{f_*}$$

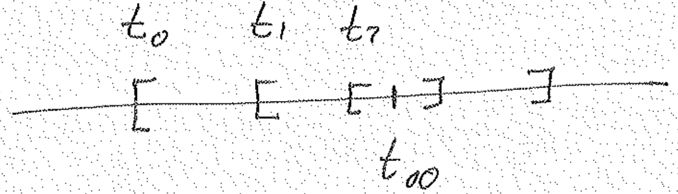
<u>And</u> Similar phenomena happen in higher dim.

<u>Cor</u>: ∂Chaos codim 1 submanifold.

"Topological class = submanifold"

Explanation parameter Universality

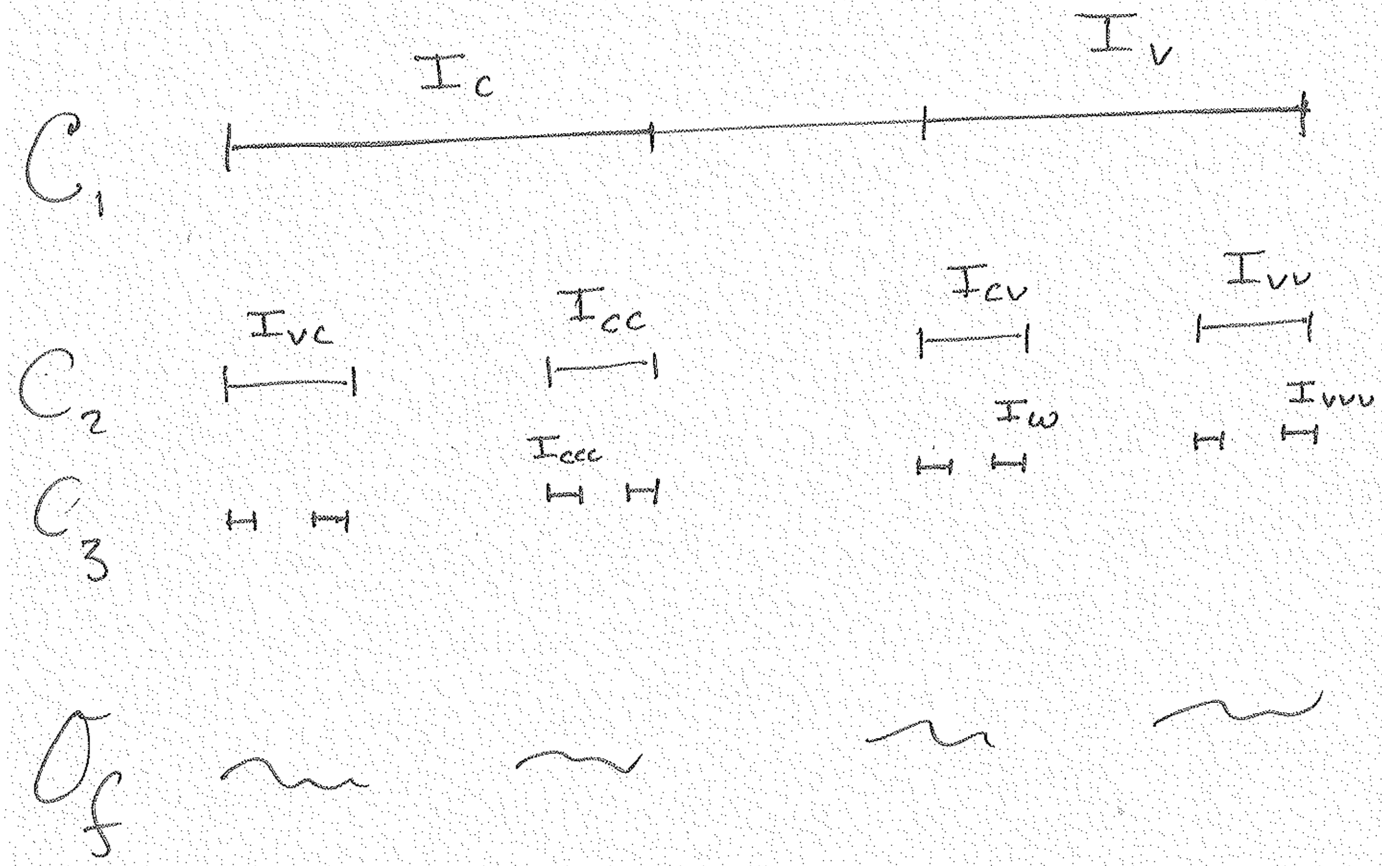




$$t_n \longrightarrow t_\infty \quad (\delta = 4.669).$$

Rmk: PD bif are not the same as the boundaries of $\mathcal{U}_n$. But essentially the same.

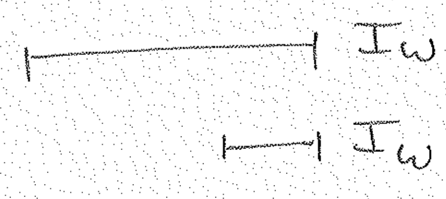
Explanation Phase Space Universality



<u>Definition</u>: Scaling ratios (σ_w)

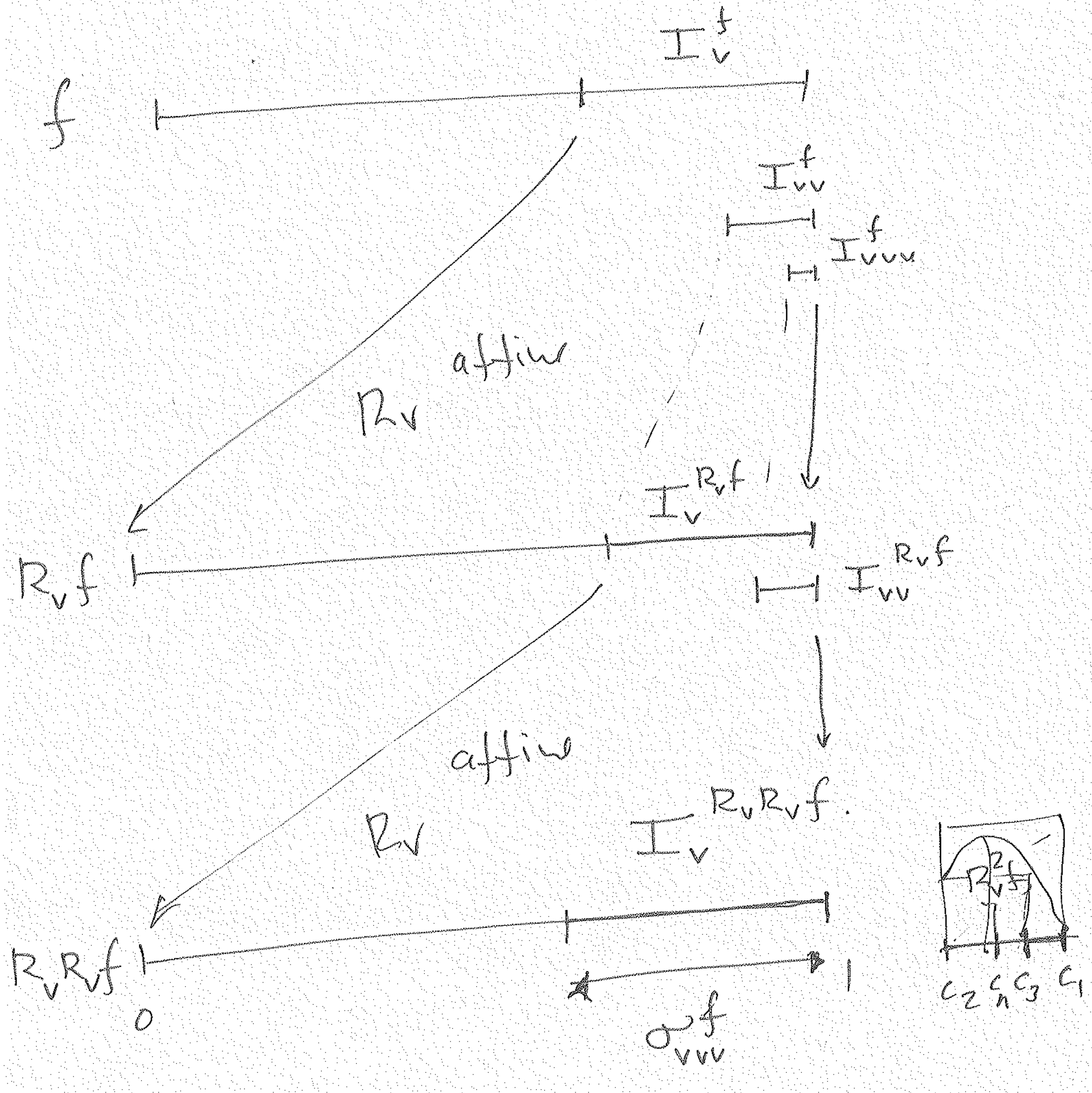
$\forall\, I_w \in C_n \quad \exists!\, I_{w'} \in C_{n-1}$ with

$I_w \subset I_{w'}$



$$0 < \sigma_w = \frac{|I_w|}{|I_{w'}|} < 1$$

Example: How to find σ^f_{vvv}?



$R_{c}, R_{v} : W^{s}_{f_*} \longrightarrow W^{s}_{f_*}$ Contraction.

So $f, g \in W^{s}_{f_*} = \partial\,\text{Chaos}$

$\Longrightarrow d(R^{n}_{v} f, R^{n}_{v} g) \xrightarrow{\text{exp.}} 0$

$\Longrightarrow \Big| \sigma^{f}_{\underbrace{v \cdots v}_{n}} - \sigma^{g}_{\underbrace{v \cdots v}_{n}} \Big| \xrightarrow{\text{exp.}} 0$

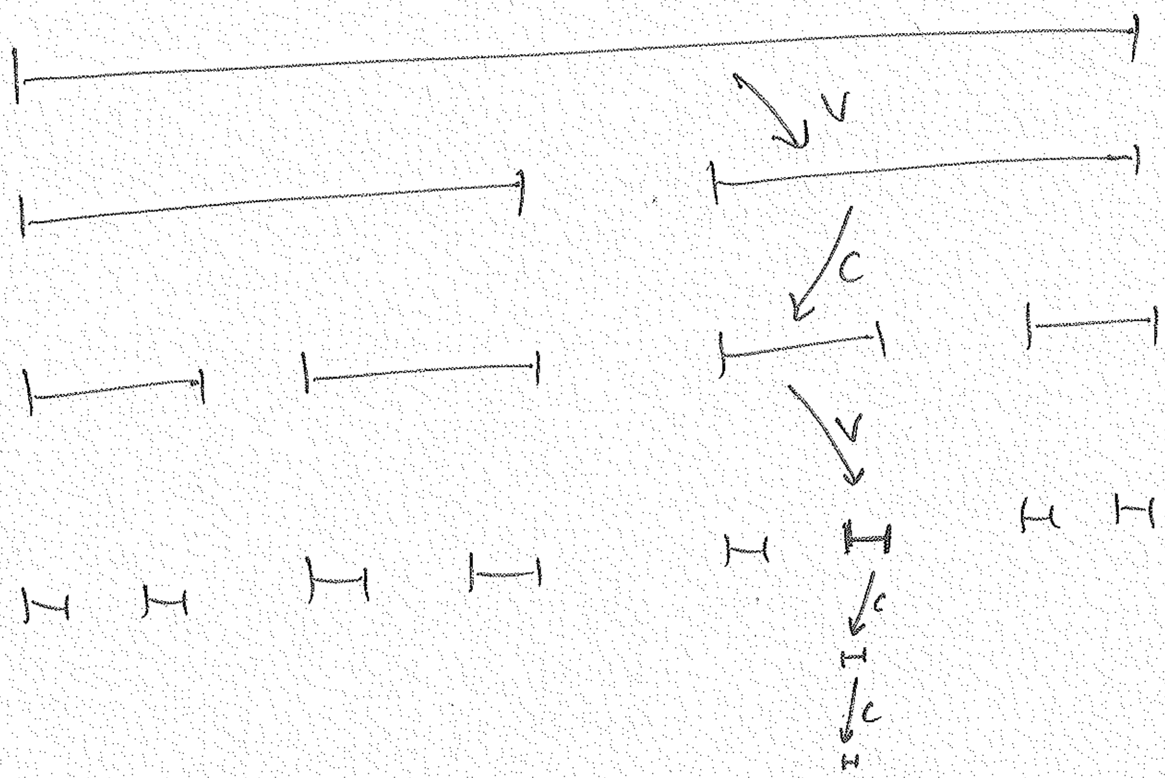
$\Longrightarrow$ <u>Phase Space Universality</u>

Similarly we can calculate any σ_{w}.

<u>Example</u>: $w = w_n w_{n-1} \cdots w_1 w_0$, $w_i \in \{c, v\}$

$$R_w f \underset{\text{def}}{=} R_{w_n} R_{w_{n-1}} \cdots R_{w_1} R_{w_0} f.$$

($|w| = n$ length of w)



$$\sigma^{f}_{ccvcv} = \sigma_c^{R_{ccvcv} f}$$

R_c, R_v are contractions on $W^s_{f_*}$.

$$\mathrm{dist}(R_w f, R_w g) = O(\rho^{|w|})$$

$$\left| \sigma^{f}_{ccvcv} - \sigma^{g}_{ccvcv} \right| = O(\rho^4)$$

$$\Sigma_f = \bigcap_{n \geq 0} \overline{\bigcup_{m \geq n} \{ \sigma_w \mid I_w \in \mathcal{C}_m \}}$$

$$= \{ \text{limits of scaling ratios} \}.$$

<u>Thm</u> (BMT)

$$\Sigma_f = \Sigma_{f_*}$$

Σ_{f_*} is a Cantor set.

O_{f_*} very far from $1/3$ Cantor set

<u>Thm</u> ~~CT conje~~

R is hyperbolic

(Renormalization Conj holds)

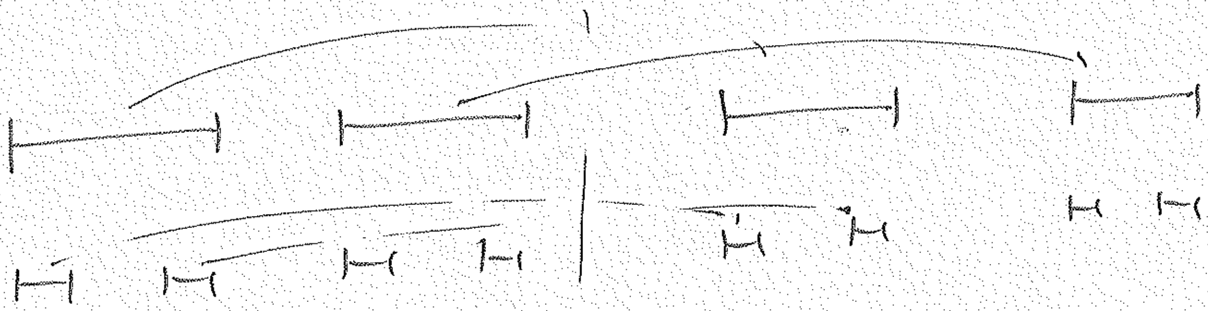
(80) Landford, <u>Sullivan</u>, <u>McMullen</u>, <u>Lyubich</u>, Avila (2009)

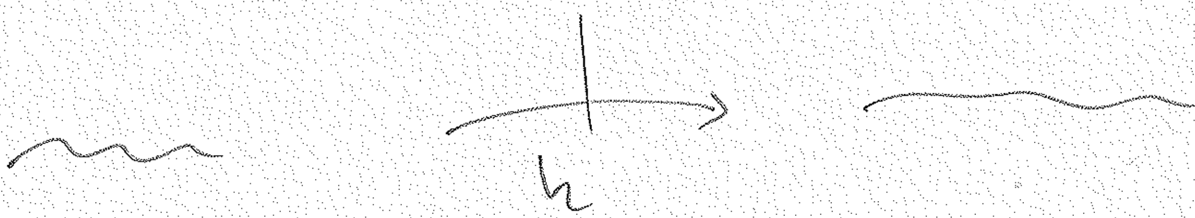
de Melo, de Faria, M.

Rigidity

$f, g \in \partial\text{Chaos} \Longrightarrow$

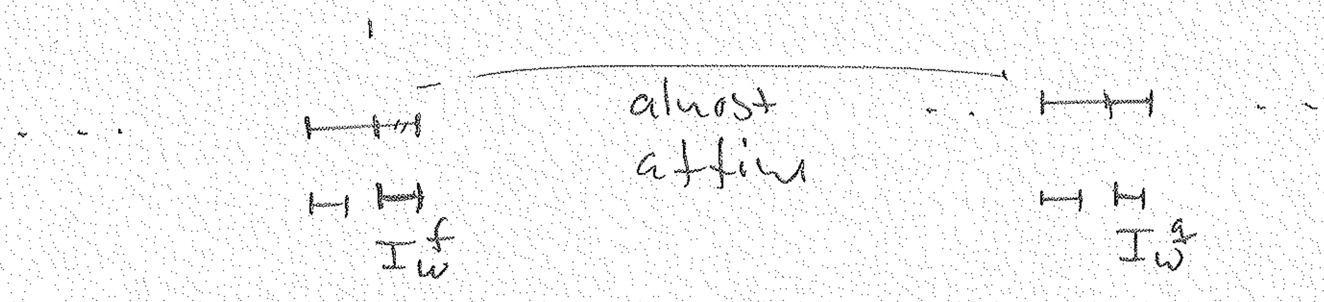
$\exists\ h : O_f \to O_g$. top Conj





Thm (Rigidity)

h is $C^{1+\alpha}$



Dyn at ∂Chaos

Plan: Renormalization explains

- Topology (PD)
- Geometry (Universality/Rigidity)
- Measure theory (μ)
- parameter dependence ($\delta = 4.66\ldots$).
  and ∂Chaos codim $= 1$.