

Short Course on Floer homology and applications

Joa Weber
IME USP

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This short course consists of five parts. In part 1 we explain the history of the Arnold conjecture and also the history of its solution. General references are [HZ94, MS95, L04]; for some of the exercises it is useful to consult in addition [GP74, MZ05].

In part 2 we discuss the finite dimensional toy model for Floer homology – the Morse-Witten complex associated to a Morse function f on a closed Riemannian manifold M . It nicely illustrates the main steps in the construction of Floer homology and provides many useful pictures to keep in mind. As a corollary we obtain the Morse inequalities: The number of critical points of f is bounded below by the sum of the Betti numbers of M . We will present the dynamical systems approach of [W06b]; see [Sch93] for a global analysis approach which more closely parallels the analytic setup of Floer homology.

In part 3 we define Floer homology for a closed symplectic manifold (M, ω) and show that it is naturally isomorphic to singular homology of M . For simplicity we assume that (M, ω) is symplectically aspherical, i.e. $[\omega]$ vanishes on $\pi_2(M)$. To confirm the Arnold conjecture for these manifolds is then a simple corollary. Our main references are [HZ94, S99].

In part 4 we introduce Floer homology for a class of noncompact, but exact (hence aspherical), symplectic manifolds, namely cotangent bundles T^*M . It is naturally isomorphic to singular homology of the free loop space of M . We indicate three methods of this calculation, see [W05], and present some details of the heat flow method. The main analytical difficulty is to deform Floer's elliptic PDE on T^*M into the parabolic heat equation on the closed Riemannian manifold M keeping track of the solutions. Here we meet

the Morse-Witten complex of lecture 1 again, but this time the manifold is the infinite dimensional free loop space ΛM and the Morse function is the classical action functional whose critical points are (perturbed) closed geodesics in M . Flow lines are provided by the solutions to the heat equation. At this point we have identified Floer homology with this version of Morse homology [W10]. To actually calculate the latter we profit from the fact that many tools from finite dimensional dynamical systems are still available for parabolic PDE's [H81]. On the other hand, having only a *semiflow* on ΛM requires new methods to construct a suitable Morse filtration – the key object to obtain a natural isomorphism to singular homology. Here Conley index theory enters.

In part 5 we shall explain the main ideas behind symplectic homology. We discuss the application in [W06a], namely existence of noncontractible periodic orbits for any compactly supported Hamiltonian on $(T^*M, -d\lambda)$ which is sufficiently large over the zero section (assuming that M is not simply connected).

The references below will be used throughout the lectures. Some of them are surveys providing references to the original research papers. Some of our references are available on www.math.sunysb.edu/~joa.

References

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