Minicurso: Transition to chaos in low dimensional dynamics
Número de horas: 8 horas
Ministrante: Charles Tresser
Filiação: IBM - EUA
Programa resumido:

1. The theme

This course is about an aspect of the mathematics of dynamical systems, but with a view both
- On applications, in particular to the Physics of the boundary of chaos and
- On Mathematics issues that may seem quite remote from the description of the boundary of chaos

when it comes to the motivations for the choice of the items being treated and the issues being raised. Some of the questions that are received full or at least extensive treatment in the recent mathematical literature will be only quickly discussed with pointers to the literature (some in depth treatment may also be given in other lectures that are more focused on a particular issue that is relevant for understanding the boundary of chaos). Rather we will insist on some aspects lesser known of the overall problem and on lesser known aspects of the most popular issues. In particular we will raise questions about whether rigidity and universality can be linked to quite different issues in dynamical systems theory such as:

- Does the Closing Lemma holds true in C^2 and higher smoothness? [8]

- Can arbitrarily smooth counter-example to the Seifert Conjecture be built out of a diffeomorphism of a two-dimensional annulus with an orbit going form the outer bounding circle to the inner bounding circle. [71], [39], [38].

REMARK: Because of the extent of material being covered, that goes over many lines of work on dynamics and beyond, only selected results will be given with a hint for a proof, with even less proofs given in full.

2. Dynamical systems with prescribed structures: the basic definitions and results and the toolkit.

We will first define dynamical systems in general [8], [10] and define the subset of the whole question that will be our framework. Basic definitions and results that will be needed will be discussed and basic tools will be explored. To provide a better view of where the main matter that we consider fits in Mathematics and in Natural Sciences, some historical perspective will be given (biased by the lecturer's own views). While we concentrate on real dynamics, we will keep a view on complex analytic methods. We will also see examples of the hiatus between the structure (that seems) natural to formulate a problem and the structure where the solution can be (easier) found. See the discussion of entropy in [22] and the literature cited there.

3. Formulating the main questions and restricting the problem to a simpler first context.

We will first discuss discuss options and choose a definition of chaos that makes the problem of the boundary a sensible issue [1], [3], [5], concept that in particular lead to a possibly reasonable boundary between regular behavior and chaotic behavior. We will discuss the role of smoothness and dimension [60] in getting a good problem.

We will then isolate non-conservative:
- (1) Maps in one dimension, [20]
- (2) Homeomorphisms and diffeomorphisms in two dimensions, [8]
- (3) Flows in three dimensions,

as the context where one can hope to get a reasonably complete answer. We will also discuss the question of the smoothness classes that make the problem richer and/or more relevant for applications.

4. Some pieces of the boundary of chaos.

This is the part that justifies the title of the mini-course, the *plat de resistance*. People who have some experience will recognize that the list that follows implies the review of a great variety of techniques, be they proper to dynamical systems theory or more generally useful in Mathematics. This part will occupy as much as possible of the time depending of the previous culture of the students following this mini-course. The precise list of topics and emphasis will be discussed with the class or at least will be decided depending on what will be known by the students (who might learn important ingredients as the course progresses from other mini-courses held in parallel).

We will discuss what is known and conjectured about:

- The transition to chaos for smooth interval maps, with fixed or unbounded number of turning points. [14], [15], [15], [16], [17], [18], [19], [20] , [21], [22], [23], [24], [26].

- The transition to chaos for circle maps and embedding of the two-dimensional annulus.

- The transition to chaos for volume contracting flows in $\mathbb{R}^3$, in particular the transition to chaos in families that create orbits bi-asymptotic to critical points: Lorentz and Lorentz-like maps [67], [68] and Sil'nikov's theorem and its applications as described in [69], [72], [73] and in [74].

Of course will review, but only at rather high level, some of the main facts and questions about Universality, Rigidity, and Renormalization [20], [76], [50] and references therein.

We will end that class by a list of problems, some of which may be described earlier during the class.

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